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The Maximum Hamilton Path Problem with Parameterized Triangle Inequality

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ABSTRACT

Given a complete graph with edge-weights satisfying parameterized triangle inequality, we consider the maximum-Hamilton path problem and design some approximation algorithms.

Keywords: Maximum Traveling Salesman Problem; Parameterized Triangle Inequality; Approximation Algorithm

1. Introduction

Routing design problems are of a major importance in-computer communications and combinatorial optimization. The traveling salesman problem (TSP) and related Hamilton path problem play important role in routing design problems. Recently, some novel approximation algorithms and randomized algorithms are designed for these problems.

Let $G=(V,E,w)$ bea complete graph in which the edge weights satisfy $w(uv)\le\gamma\cdot(w(ux)+w(xv))$ for all distinct nodes $u,x,v\in V$, the maximum Hamilton path problem(MHP) with γ -parameterized triangle inequality is to find a path with maximum weight that visits each node exactly once.

A cycle cover is a subgraph in which each vertex in *V* has a degree of exactly 2. A maximum cycle cover is one with maximum total edge weight which can be computed in $O(n^3)$ time [4]. It is well-known that the weight of the maximum cycle cover is an upper bound on OPT, where OPT denotes the weight of an optimal Hamilton path (or tour). All the algorithms mentioned in this paper start by constructing a maximum-weight cycle cover. A subtour in this paper is a subgraph with no non-Hamiltonian cycles or vertices of degree greater than 2. Thus by adding edges to a subtour it can be completed to a tour. In this paper, for asubset $E'\subseteq E$, $w(E')$ denotes the (expected) total weight of the edges in E'.

We call that an algorithm is a ρ -approximation algorithm for the MHP if it always produces a feasible solution with objective at least ρOPT, where OPT denotes the optimal value.

When $\gamma=1$, Monnot [10] presented a $1/2$ -approximation algorithm for the MHP with two specified endpoints and a $2/3$ -approximation algorithm for the MHP with one specified end point. To the best of knowledge, this is the unique result about MHP.

In this paper, we first present a deterministic approxi-Mation algorithm with performance ratio $\frac{4\gamma+1}{6\gamma}-\frac{1}{2n\gamma}$ For the MHP without specified endpoint, and a random-Ized algorithm with expected ratio $\frac{3\gamma+\frac{1}{2}}{4\gamma}-O\left(\frac{1}{\sqrt[3]{n}}\right)$ by Modifying the algorithm in [7], where *n* is the number of vertices in *G*. We also present a determinant approximation algorithm with performance ratio $\frac{4\gamma+1}{6\gamma}$ for the M H P with one specified end point, which improves the result in [10].

A closely related problem is the maximum traveling salesman problem (MTSP) with γ -parameterized triangleine quality which is to find a tour with maximum weight that visits each node exactly once in a complete graph in which the edge weights satisfy γ -parameterized triangle in equality. Zhang, Yin, and Li [14] introduced the MTSP with γ -parameterized triangle inequality for

$\gamma\in[\frac{1}{2},1)$, motivated by the minimum traveling salesman problem with γ -parameterized triangle inequality in [13]. They firstcompute a maximum-weight cycle cover $\mathcal{C}=\{C_1,,\ldots,C_l\}$, and then delete the minimum-weight of the edges in C_i and extend the remained paths to a

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Hamiltonian cycle. They [14] proved that the performance ratio of their algorithm is $\frac{K\gamma+1-2\gamma}{K\gamma}$, where $K=\min\{|C_i| \mid i=1,2,\ldots,l\}$. Since $|C_i|\ge 3$ for the cycle cover of undirected graph, the ratio is at least $\frac{\gamma+1}{3\gamma}$.

Notice that the MTSP wit γ h -parameterized triangle inequality is a special case of the maximum TSP which is first considered by Fisher, Nemhauser and Wolsey [3], where two approximation algorithms are given. In 1984, Serdyukov[12] presented (in Russian) an elegant $\frac{3}{4}$- approximation algorithm. Afterwards, Has sin and Rubinstein [6] presented a randomized algorithm whose expected approximation ratio is at least $\frac{25(1-\varepsilon)}{33-32\varepsilon}$, where ε is an arbitrarily small constant. Chen, Okamoto, and Wang [2] first gave a deterministic approximation algorithm with the ratio better than $\frac{3}{4}$, which is a $\frac{61}{81}$- approximation and a nontrivial derandomization of the algorithm from [6]. Very recently, Paluch, Mucha, and Madry [11] first presented a fast deterministic $\frac{7}{9}$-approximation algorithm for the maximum TSP, which isthe currently best result.

If $\gamma=1$, the MTSP with γ-parameterized triangle in equality is exactly the metric maximum TSP. Kostochka and Serdyukov [7] first designed a $\frac{5}{6}$-approximation algorithm for the metric maximum TSP. Hassin and Rubinstein [5]designed a randomized approximation algorithm for themetric maximum TSP whose expected approximation ratio is $\frac{7}{8}-O\left(\frac{1}{\sqrt[3]{n}}\right)$, where *n* is the number of vertices in the graph. This algorithm had later been derandomized by Chen and Nagoya [1], at a cost of a slightly worse approximation factor of $\frac{7}{8}-O\left(\frac{1}{\sqrt[3]{n}}\right)$. Very recently, Kowalik and Mucha [9] extended the approach of processing local configuration susing so-called loose-ends in [8], and presented a deterministic $\frac{7}{8}$-approximation algorithm for the metric maximum TSP, which is the currently best result.

The paper is divided into four sections. In Section 2, we will present some algorithms for the MHP problem-without specified endpoint. In Section 3, we will presentan algorithms for the MHP problem with one specified end point. The paper is concluded in the last section.

2. The MHP Problem without Specified Endpoint

2.1. Modified Randomized Kostochka & Serdyukov's Algorithm

In [5], Hassin and Rubinstein gave a randomized algorithm the maximum TSP [7]. We modify it to the MHP problem.

Algorithm A0:

1. Compute a maximum cycle cover
$$\mathcal{C}=\{C_1,C_2,\ldots,C_l\}.$$
Without loss of generality, we assume that C_1 satisfies $\frac{w(C_1)}{|C_1|}\le\frac{w(C_i)}{|C_i|}$ for any $i=1,2,\ldots,l-1$.

2. Delete from each cycle a random edge. Let u_i and v_i be the ends of the path P_i that results from C_i.

3. Give each path a random orientation and form a Hamilton path *HP* by adding connecting edges between the head of P_i and thetail of P_{i+1}, $i=1,2,\ldots,l-1$.

Theorem 1. For any $\gamma\ge\frac{1}{2}$, the expected approxima-Tion ratio of Algorithm A0 for the MHP problem is
$$\left(\frac{4\gamma+1}{6\gamma}-\frac{1}{2n\gamma}\right)OPT.$$

Proof. Clearly,
$$\begin{aligned}
w(T)&=\sum_{i=1}^{l}w(P_i)+\frac{1}{4}\sum_{i=1}^{l-1}\big(w(u_i,u_{i+1})+w(u_i,v_{i+1})\\
&\qquad+w(v_i,u_{i+1})+w(v_i,v_{i+1})\big)\\
&\ge\sum_{i=1}^{l}w(P_i)+\frac{1}{2\gamma}\sum_{i=1}^{l}w(u_i,v_i)-\frac{1}{2\gamma}w(u_1,v_1)\\
&=\sum_{i=1}^{l}w(C_i)+\left(\frac{1}{2\gamma}-1\right)\sum_{i=1}^{l}w(u_i,v_i)-\frac{1}{2\gamma}w(u_1,v_1)\\
&\ge\left(1+\frac{1}{K}\left(\frac{1}{2\gamma}-1\right)-\frac{1}{2n\gamma}\right)w(\mathcal{C})\\
&\ge\left(\frac{4\gamma+1}{6\gamma}-\frac{1}{2n\gamma}\right)OPT.
\end{aligned}$$

Here, the first inequality follows from the γ-parameterized triangle inequality, the second inequality follows from the assumption that C_1 satisfies $\frac{w(C_1)}{|C_1|}\le\frac{w(C_i)}{|C_i|}$, and the last inequality follows from $K\ge 3$. This completes the proof. It is well-known that the above randomized version of Kostochka & Serdyukov's algorithm can easily be derandomized by the standard method of conditional expectation.

2.2. Modified Hassin & Rubinstein's Algorithm

Based on Serdyukov's algorithm in [12] and the randomization version of Kostochka & Serdyukov's algorithm, Hassin and Rubinstein [5] presented a new algorithm for the maximum TSP problem. We modify it to the MHP problem.

Algorithm B0:

1) Compute a maximum-weight cycle cover $\mathcal{C}=\{C_1,C_2,\ldots,C_l\}$ of *G*.

2) Compute a maximum-weight matching *M* in *G*.

3) For $i=2,\ldots,s$, identify $e,f\in E\cap C_i$ such that both $M\cup\{e\}$ and $M\cup\{f\}$ are subtours. Randomly choose $g\in\{e,f\}$ (each with probability $1/2$). Set $P_i=C_i-\{g\}$ and $M=M\cup\{g\}$.

4) Complete $\bigcup_{i=1} P_i$ into a tour T_1 as in Kostochka & Serdyukov's algorithm [7].

5) Let *S be the* set of end nodes of paths in *M*. Compute a random perfect matching M_S over *S*. Delete an edge from each cycle in $M\cup M_S$. Arbitrarily complete $M\cup M_S$ into a tour T_2.

6) Let *T* be the maximum-weighted tour between T_1 and T_2 and *e* be the lightest edge in *T*, output the Hamilton path $HP=T-\{e\}$.

Theorem 2. For any $\gamma\in[\frac{1}{2},+\infty)$, the expected weight of the tour *T* returned by the Hassin & Rubinstein's algorithm[5] for the maximum TSP with γ-parameterized triangle inequality satisfies

$$w(T)\geq\left(\frac{3\gamma+\frac{1}{2}}{4\gamma}-O\left(\frac{1}{\sqrt{n}}\right)\right)OPT.$$

Proof. Denote by α the relative weight in *C* of the edges that were candidates for deletion. Let OPT' denote the value of the maximum-weighted Hamilton cycle. Clearly, $OPT'\geq OPT$. Similarly to the proof of Theorem 1 in [5], we obtain

$$w(T_1)\geq\left(1-\frac{\alpha}{2}+\frac{\alpha}{2}\cdot\frac{1}{2\gamma}\right)w(\mathcal{C})\geq\frac{4\gamma+(1-2\gamma)\alpha}{4\gamma}OPT'.$$

Also, we have

$$w(M_S)\geq\frac{1-\alpha}{4\gamma}w(\mathcal{C})$$

$$w(M\cup M_S)\geq\left(\frac{1}{2}+\frac{\alpha}{2}+\frac{1-\alpha}{4\gamma}\right)w(\mathcal{C})$$

$$=\frac{1+2\gamma-(1-2\gamma)\alpha}{4\gamma}OPT',$$

and

$$w(T_2)\geq\frac{1+2\gamma-(1-2\gamma)\alpha}{4\gamma}\left(1-O\left(\frac{1}{\sqrt[3]{n}}\right)\right)OPT'.$$

Thus,

$$\mathrm{w}(T)=\max\{\mathrm{w}(T_1),\mathrm{w}(T_2)\}$$

$$\geq\frac{\mathrm{w}(T_1)+\mathrm{w}(T_2)}{2}\geq\left(\frac{3\gamma+\frac{1}{2}}{4\gamma}-O\left(\frac{1}{\sqrt[3]{n}}\right)\right)OPT'.$$

As *e* is the lightest edge in *T*, we have

$$w(HP)\geq\left(1-\frac{1}{n}\right)\mathrm{w}(T)$$

$$\geq\left(1-\frac{1}{n}\right)\left(\frac{3\gamma+\frac{1}{2}}{4\gamma}-O\left(\frac{1}{\sqrt[3]{n}}\right)\right)OPT'$$

$$\geq\left(\frac{3\gamma+\frac{1}{2}}{4\gamma}-O\left(\frac{1}{\sqrt[3]{n}}\right)\right)OPT.$$

Based on the idea of Chen et al. [2] and the propertiesof a folklore partition of the edges of a $2n$-vertex complete undirected graph into $2n-1$ perfect matchings, Chen and Nagoya [1] derandomize Hassin & Rubinstein's algorithm mat a cost of a slightly worse approximation factor of $\frac{7}{8}-O\left(\frac{1}{\sqrt[3]{n}}\right)$. Similarly to [1], we obtain the following theorem.

Theorem 3. For any $\gamma\geq\frac{1}{2}$, there is a deterministic Algorithm for the maximum TSP with γ-parameterized triangle inequality with approximation ratio of

$$\frac{3\gamma+\frac{1}{2}}{4\gamma}-O\left(\frac{1}{\sqrt[3]{n}}\right).$$

3. The Mhp Problem with One Specified Endpoint

Let i be the specified endpoint. The MHP problem with one specified endpoint is to find a maximum- weighted Hamilton path starting from *s*. Our algorithm is based on computing cycle cover containing the special edge (s,r) for each $r\in V-\{s\}$. We find $n-1$ feasible solutions and output the best one. The detailed algorithms are presents as follows.

Algorithm A1:

1) For each $r \in V-\{s\}$, compute a maximum cycle cover $\mathcal{C}^r=\{C_1^r, C_2^r, \ldots, C_{l_r}^r\}$ containing the edge (s,r). Assume that C_1^r contains (s,r) and $w(s,r)=0$.

2) Delete a minimum weight edge from each cycle C_i^r, for $i=2,\ldots,l_r$, and the edge (s,r) from C_1^r. Let u_i and v_i bethe ends of the path P_i that results from C_i, where $u_1=s$ and $v_1=r$.

3) If $l_r=2$, construct two Hamilton paths as follows:

$$HP_1 : sP_1ru_2P_2v_2,$$
$$HP_2 : sP_1rv_2P_2u_2.$$

If $l_r \geq 3$, construct four Hamilton paths as follows.

$$HP_1 : sP_1ru_2P_2v_2u_3P_3v_3\cdots u_{l_r}P_{l_r}v_{l_r},$$
$$HP_2 : sP_1rv_2P_2u_2v_3P_3u_3\cdots v_{l_r}P_{l_r}u_{l_r}.$$

When *l* is odd,

$$HP_3 : sP_1rv_2P_2u_2u_3P_3v_3\cdots u_{l_r}P_{l_r}v_{l_r},$$
$$HP_4 : sP_1ru_2P_2v_2v_3P_3u_3\cdots v_{l_r}P_{l_r}u_{l_r}.$$

When *l* is even,

$$HP_3 : sP_1rv_2P_2u_2v_3P_3u_3\cdots v_{l_r}P_{l_r}u_{l_r},$$
$$HP_4 : sP_1ru_2P_2v_2u_3P_3v_3\cdots u_{l_r}P_{l_r}v_{l_r}.$$

4) Compute a maximum-weighted Hamilton path HP^r in $\{HP_i | i=1,2,3,4\}$ where $HP_3=HP_4=\varnothing$, if $l_r=2$.

5. Output a maximum-weighted Hamilton path *HP* in $\{HP^r | r\in V-\{s\}\}$.

Lemma 4. For each $r\in V-\{s\}$, the objective value of HP^r is at least $\frac{4\gamma+1}{6\gamma}w(\mathcal{C}^r)$.

Proof. If $l_r=2$, we have

$$\begin{aligned}
&w(HP_1)+w(HP_2)\\
&=2w(P_1)+2w(P_2)+w(r,u_2)+w(r,v_2)\\
&\geq 2w(P_1)+2w(P_2)+\frac{1}{\gamma}w(u_2,v_2)\\
&\geq 2w(\mathcal{C}^r)+\left(\frac{1}{\gamma}-2\right)w(u_2,v_2)\\
&\geq 2w(\mathcal{C}^r)+\left(\frac{1}{\gamma}-2\right)\frac{w(C_2^r)}{|C_2^r|}\\
&\geq \frac{4\gamma+1}{3\gamma}w(\mathcal{C}^r).
\end{aligned}$$

Thus,

$$w(HP^r)\geq \frac{w(P_1)+w(P_2)}{2}\geq \frac{4\gamma+1}{6\gamma}w(\mathcal{C}^r).$$

If $l_r \geq 3$, we have

$$\begin{aligned}
&\sum_{i=1}^{4}w(HP_i)\\
&=4\sum_{i=1}^{l_r}w(P_i)+2w(r,u_2)+2w(r,v_2)\\
&\quad+\sum_{i=2}^{l_r-1}\big(w(u_i,u_{i+1})+w(u_i,v_{i+1})+w(v_i,u_{i+1})+w(v_i,v_{i+1})\big)\\
&\geq 4\sum_{i=1}^{l_r}w(P_i)+\frac{2}{\gamma}\sum_{i=2}^{l_r}w(u_i,v_i)\\
&=4w(\mathcal{C}^r)+\left(\frac{2}{\gamma}-4\right)\sum_{i=2}^{l_r}w(u_i,v_i)\\
&\geq \frac{8\gamma+2}{3\gamma}w(\mathcal{C}^r),
\end{aligned}$$

where the second inequality follows from $\gamma\geq\frac{1}{2}$, and the last inequality follows from $K\geq 3$ and $w(\mathcal{C})\geq OPT$. Therefore,

$$w(HP^r)\geq \frac{1}{4}\sum_{i=1}^{4}w(HP_i)\geq \frac{4\gamma+1}{6\gamma}w(\mathcal{C}^r).$$

This completes the proof.

Theorem 5. The weight of the Hamilton path *HP* is at least $\frac{4\gamma+1}{6\gamma}OPT$.

Proof. Since *HP* is the maximum-weighted Hamilton pathin $\{HP^r | r\in V-\{s\}\}$, we have

$$\begin{aligned}
w(HP)&\geq \max_{r\in V-\{s\}} w(HP^r)\\
&\geq \max_{r\in V-\{s\}}\frac{4\gamma+1}{6\gamma}w(\mathcal{C}^r)\\
&\geq \frac{4\gamma+1}{6\gamma}OPT,
\end{aligned}$$

where the second inequality follows from Lemma 4, and thelast inequality follows from the fact $\max_{r\in V-\{s\}} w(\mathcal{C}^r)$ isan obvious upper bound on OPT.

4. Conclusions

In this paper, we presented some approximation algorithms for two variants of the maximum Hamilton path problem with parameterized triangle inequality. It is interesting to design some better approximation algorithms for these problems.

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