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Existence for a Higher Order Coupled System of Korteweg-de Vries Equations

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Abstract

Consider the following system of coupled Korteweg-de Vries equations, $\begin{cases}\Delta^2 u+\lambda_1 u=u^2+\beta uv^2,\\ \Delta^2 v+\lambda_2 v=v^2+\beta u^2 v.\end{cases}$ where $u,v\subseteq W^{2,2}$, $2\le N\le 7$ and $\lambda_i,\beta>0$, β denotes a real coupling parameter. Firstly, we prove the existence of the solutions of a coupled system of Korteweg-de Vries equations using variation approach and minimization techniques on Nehari manifold. Then, we show the multiplicity of the equations by a bifurcation theory which is rare for studying higher order equations.

Keywords

System of Korteweg-de Vries Equations, Normalized Vector Solitary Waves, Variation Approach

1. Introduction

It is well known that the form of the coupled nonlinear Korteweg-de Vries equations is as follows

$$\begin{cases}\phi_t+\phi_{xxx}+P(\phi,\psi)_x=0,\\ \psi_t+r\psi_x+\psi_{xxx}+Q(\phi,\psi)_x=0,\end{cases}\tag{1.1}$$

where r is a real constant, $\phi=\phi(x,t),\psi=\psi(x,t)$ are real-valued functions of x and t, $\beta>0$ is a coupling parameter and P, Q satisfy $P(\phi,\psi)=H_u(\phi,\psi),Q(\phi,\psi)=H_v(\phi,\psi)$ for a small function H. Model represents the physical problem of describing the strong interaction of two-dimensional long internal gravity waves propagating on neighboring pycnoclines in a stratified fluid. In this paper we consider a special case of (1.1), namely the following system of nonlinear evolution equations:

$$\begin{cases} \phi_t + \phi_{xxx} + \left(uv^2\right)_x = 0, \\ \psi_t + r\psi_x + \psi_{xxx} + \left(u^2 v\right)_x = 0, \end{cases} \tag{1.2}$$

We always look for solutions of (1.2) of the form

$$\left(\phi(x,t), \psi(x,t)\right) = \left(u\left(x-\lambda_1 t\right), v\left(x-\lambda_2 t\right)\right), \tag{1.3}$$

where $\lambda_1, \lambda_1 \in \mathbb{R}$. Through calculating, if $r=0$ we get

$$\begin{cases} -u'' + \lambda_1 u = \beta u^p v^{p+1}, \\ -v'' + \lambda_2 v = \beta u^{p+1} v^p, \end{cases} \tag{1.4}$$

Now we consider it in higher dimensional cases, as follows:

$$\begin{cases} \Delta^2 u + \lambda_1 u = u^2 + \beta u v^2, \\ \Delta^2 v + \lambda_2 v = v^2 + \beta u^2 v, \end{cases} \tag{1.5}$$

where $u, v \subseteq W^{2,2}$, $2 \le N \le 7$ and $\lambda_i, \beta > 0$, β is a coupling parameter.

The system of (1.1) has been analysed many times. For example, see the recently derived model by Gear and Grimshaw [1], considering

$$H(u,v) = \frac{b_2}{6} u^3 + \frac{1}{6} v^3 + \frac{a_2 b_2}{2} u^2 v + \frac{a_1 b_2}{1} u v^2, \tag{1.6}$$

where a_1, a_2, b_1, b_2 are constants. Moreover, the system (1.2) has been extensively studied in recent years and is also a special case of a general class of nonlinear evolution equations considered in [2] in the inverse scattering context. More properties of the system (1.2) have been proved. Alarcon and Montenegro proved the local and global well-posedness results for the initial-value problem for (1.2) with $r=0, p=1$ in [3] and [4]. Panthee and Scialom improved the well-posedness results obtained in the case when $p=2$ in [2].

As we know, many analyses about higher order equation have been done many years ago including the third and fifth order KdV equation. Firstly, it is already well known that the third order KdV equation describes the evolution of weakly nonlinear and weakly dispersive shallow waves in physical contexts such as plasma, ion-acoustic waves, stratified internal, and atmospheric waves and it has been analysed during the last decades. For the fifth order equation, the results are less than the third. But it has attracted increasing attentions (see [5]-[22]) and is used to model many physical phenomena such as gravity-capillary waves on a shallow layer and magnetosome propagation in plasmas. For example, Baker took the work and published in 1903; Li Xiaofeng proved the existence of solitary wave solutions of fifth order KdV equations in recent years. Santosh Bhattarai proved the existence of travelling-wave solutions of coupled KdV equations when it loses the compactness, using the method of concentrate compactness principle of Lions in 2015.

We know that the system of higher order equations is rare. We only can find other similar fourth-order systems studying the interaction of the long and short waves have appeared. P. Lvarez-Caudevilla and E. Colorado researched the coupled nonlinear Schrodinger Equations (1.7) and the system of Schrodinger and Korteweg-de Vries Equations (1.8).

$$\begin{cases}\Delta^2 u+\lambda_1 u=\mu_1 u^3+\beta u v^2,\\ \Delta^2 v+\lambda_2 v=\mu_2 v^3+\beta u^2 v,\end{cases} \tag{1.7}$$

and

$$\begin{cases}\Delta^2 u+\lambda_1 u=u^3+\beta u v,\\ \Delta^2 v+\lambda_2 v=\frac{1}{2}|v|^2 v+\beta u^2,\end{cases} \tag{1.8}$$

They proved the existence of equations using the variation approach and minimization techniques on Nehari manifold and the multiplicity of the equations by fibering map.

But we know that there is not previous mathematics work analyzing a higher order system as (1.3) and we get the multiplicity of the equations by a bifurcation theory which is not founded in other higher order equations article.

We organize the paper as follows. In Section 2, we introduce the notation, establish the functional framework, define the Nehari manifold and give the main theorem. In Section 3, we construct semi-trivial solutions and show the properties depending on the coupling parameter. Moreover, we devoted to proving the main results of the paper by the variation principle and mountain-pass theorem. In Section 4, using the Crandall-Rabinowitz local bifurcation theory, we show the multiplicity of the ground state solutions.

2. Preliminaries and Main Theorems

In $H^2(\mathbb{R}^N)$, we define the following equivalent norm and scalar product:

$$\langle u,v\rangle_i:=\int_{\mathbb{R}^N}(\Delta u\cdot\Delta v+\lambda_i uv),\quad \|u\|_i^2:=\langle u,u\rangle_i,\ i=1,2, \tag{2.1}$$

Accordingly, the inner product and induced norm on $H^2\times H^2$ are given by

$$\begin{aligned}&\langle(u,v),(\xi,\eta)\rangle=\int_{\mathbb{R}^N}(\Delta u\cdot\Delta\xi+\Delta v\cdot\Delta\eta+\lambda_1 u\xi+\lambda_2 v\eta),\\ &\|(u,v)\|^2=\|u\|_1^2+\|u\|_2^2,\end{aligned} \tag{2.2}$$

We define $\mathbb{E}$ the radially symmetric functions in $H^2(\mathbb{R}^N)$ and $\mathbb{H}=H\times H$. In addition, we define the energy functional associated with system (1.5) by

$$\Phi(\mathbf{u})=\frac{1}{2}\|u\|_1^2+\frac{1}{2}\|v\|_2^2-\frac{1}{3}\int_{\mathbb{R}^N}(u^3+v^3)-\frac{1}{2}\beta\int_{\mathbb{R}^N}u^2v^2 \tag{2.3}$$

and

$$I_1(u)=\frac{1}{2}\|u\|_1^2-\frac{1}{3}\int_{\mathbb{R}^N}u^3,\quad I_2(v)=\frac{1}{2}\|v\|_1^2-\frac{1}{3}\int_{\mathbb{R}^N}v^3,$$

are the energy functionals of the uncoupled equations. Then, we define

$$\Psi(\mathbf{u})=\Phi'(\mathbf{u})[\mathbf{u}]=\|\mathbf{u}\|^2-\int_{\mathbb{R}^N}(u^3+v^3)-2\beta\int_{\mathbb{R}^N}u^2v^2. \tag{2.4}$$

Now, we restrict the Nehari Manifold to the setting, denoting it as

$$\mathcal{N}=\{\mathbf{u}\in\mathbb{E}\setminus\{(0,0)\}:\Psi(\mathbf{u})=0\}.$$

Remark 2.1. (*see* [23] [24] [25])

Let

$$2^* = \begin{cases} \dfrac{2N}{N-4}, & \text{if } N > 4, \\ \infty, & \text{if } 1 \le N \le 4. \end{cases}$$

Then we have the following Sobolev embedding:

$$H^2\left(\mathbb{R}^N\right) \hookrightarrow L^p\left(\mathbb{R}^N\right), \quad \text{for} \begin{cases} 2 \le p \le 2^*, & \text{if } N \ne 4, \\ 2 \le p < 2^*, & \text{if } N = 4. \end{cases}$$

Proposition 2.1. *We are going to prove some properties for* $\mathcal{N}$ *and* Φ *on* $\mathcal{N}$.

1) $\mathcal{N}$ *is a locally smooth manifold.*

2) $\mathcal{N}$ *is a complete metric space.*

3) $\mathbb{E}\setminus\{(0,0)\}$ *is a critical point of* Φ *if and only if* $\mathbf{u}$ *is a critical point of* Φ *constrained on* $\mathcal{N}$.

4) Φ *is bounded from below on* $\mathcal{N}$.

Proof. 1) Differentiating expression (2.4) yields

$$\Psi'(\mathbf{u})[\mathbf{u}] = 2\|\mathbf{u}\|^2 - 3\int_{\mathbb{R}^N}\left(u^3+v^3\right) - 8\beta\int_{\mathbb{R}^N} u^2v^2, \tag{2.5}$$

and because of $\forall \mathbf{u} \in \mathcal{N}$, we have the fact that $\Psi(\mathbf{u}) = 0$.

Then, we obtain

$$\Psi'(\mathbf{u})[\mathbf{u}] = \Psi'(\mathbf{u})[\mathbf{u}] - 3\Psi(\mathbf{u}) = -\|\mathbf{u}\|^2 - 2\beta\int_{\mathbb{R}^N} u^2v^2 < 0, \tag{2.6}$$

Then, $\mathcal{N}$ is a locally smooth manifold near any point $\mathbf{u} \ne \mathbf{0}$ with $\Psi(\mathbf{u}) = 0$.

2) Let $\mathbf{u}_n \subset \mathcal{N}$ be a sequence such that $\|\mathbf{u}_n - \mathbf{u}_0\| \to 0$ as $n \to +\infty$. By the embedding theorem, we have $|u_n - u_0|_p \to 0$ and $|v_n - v_0|_p \to 0$ for $2 \le p \le 2^*$. It is clear that

$$\begin{aligned} &\Phi'(\mathbf{u}_n)[\mathbf{u}_n] - \Phi'(\mathbf{u}_0)[\mathbf{u}_0] \\ &= \int_{\mathbb{R}^N}|\Delta u_n|^2 - |\Delta u_0|^2 + \int_{\mathbb{R}^N}\lambda_1\left(u_n^2 - u_0^2\right) + \int_{\mathbb{R}^N}|\Delta v_n|^2 - |\Delta v_0|^2 + \int_{\mathbb{R}^N}\lambda_1\left(v_n^2 - v_0^2\right) \\ &\quad - \int_{\mathbb{R}^N}\left(u_n^3 - u_0^3\right) - \int_{\mathbb{R}^N}\left(v_n^3 - v_0^3\right) - 2\beta\int_{\mathbb{R}^N}\left(u_n^2v_n^2 - u_0^2v_0^2\right). \end{aligned} \tag{2.7}$$

Since $|u_n - u_0|_2 \to 0$ and $|v_n - v_0|_2 \to 0$, applying Holder's inequality, we get

$$\begin{aligned} &\left|\int_{\mathbb{R}^N}\left(u_n^2v_n^2 - u_0^2v_0^2\right)\right| \\ &\le \int_{\mathbb{R}^N} u_n^2\left|v_n^2 - v^2\right| + \int_{\mathbb{R}^N} v^2\left|u_n^2 - u^2\right| \\ &\le |u_n|_4^2\left(\int\mathbb{R}^N\left(v_n - v_0\right)^2\left(v_n - v_0\right)^2\right)^{\frac{1}{2}} + |v|_4^2\left(\int\mathbb{R}^N\left(u_n - u_0\right)^2\left(u_n - u_0\right)^2\right)^{\frac{1}{2}} \\ &\le |u_n|_4^2 \cdot |v_n - v_0|_4 \cdot |v_n - v_0|_4 + |v|_4^2 \cdot |u_n - u_0|_4 \cdot |u_n - u_0|_4 \\ &\to 0, \end{aligned} \tag{2.8}$$

So we have $\Phi'(\mathbf{u}_n)[\mathbf{u}_n] - \Phi'(\mathbf{u}_0)[\mathbf{u}_0] \to 0$. Because of $\Phi'(\mathbf{u}_n)[\mathbf{u}_n] = 0$, we get $\Phi'(\mathbf{u}_0)[\mathbf{u}_0] = 0$. Using $\|\mathbf{u}_n\|^2 > \rho$ and $\|\mathbf{u}_n - \mathbf{u}_0\| \to 0$, we get $\mathbf{u}_n \ne (0,0)$. Hence $\mathbf{u}_n \in \mathcal{N}$ and $\mathcal{N}$ is a complete metric space.

Taking the derivative of the functional Φ in the direction $\mathbf{h} = (h_1, h_2)$, we find

$$\Phi'(\mathbf{u})[\mathbf{h}] = \int_{\mathbb{R}^N}(\Delta u h_1 + \lambda_1 u h_1 + \Delta v h_2 + \lambda_2 v h_2) - \int_{\mathbb{R}^N}(u^2 h_1 + v^2 h_2) - \beta\int_{\mathbb{R}^N}(uv^2 h_1 + u^2 v h_2).$$

The second derivative of Φ is given by

$$\Phi''(\mathbf{u})[\mathbf{h}]^2 = \|\mathbf{h}\|^2 - 2\int_{\mathbb{R}^N}(uh_1^2 + vh_2^2) - \beta\int_{\mathbb{R}^N}(u^2h_2^2 + v^2h_1^2 + 4uvh_1h_2).$$

So, we have

$$\Phi''(\mathbf{0})[\mathbf{h}]^2 = \|\mathbf{h}\|^2,$$

which is positive definite so that $\mathbf{0}$ is a strict minimum critical point of Φ. As a consequence, we have that $\mathcal{N}$ is a smooth complete manifold, and there exists a constant $\rho > 0$ such that

$$\|\mathbf{u}\|^2 > \rho, \quad \forall \mathbf{u} \in \mathcal{N}. \tag{2.9}$$

3) Assume that $(u_0, v_0) \in \mathcal{N}$ is a critical point of Φ and with $\Psi(\mathbf{u}) = \Phi'(\mathbf{u})[\mathbf{u}]$, then there is a Lagrange multiplier $\wedge \in \mathbb{R}$ such that

$$\Phi'(u_0, v_0) = \wedge\Psi'(u_0, v_0). \tag{2.10}$$

Apply both sides to (u_0, v_0) and we can get

$$0 = (\Phi'(u_0, v_0), (u_0, v_0)) \wedge (\Psi'(u_0, v_0), (u_0, v_0)). \tag{2.11}$$

Combining (2.6) and (2.11), we get $\wedge = 0$. Now (2.10) gives $\Phi'(u_0, v_0) = 0$, *i.e.* (u_0, v_0), is a critical point of Φ.

4) The functional constrained on $\mathcal{N}$ takes the form combining (2.3) and (2.4)

$$\Phi_{\mathcal{N}}(\mathbf{u}) = \frac{1}{6}\|\mathbf{u}\|^2 + \frac{1}{6}\beta\int_{\mathbb{R}^N} u^2v^2, \tag{2.12}$$

using (2.9) and (2.12), we can get

$$\Phi(\mathbf{u}) \ge \frac{1}{6}\rho, \quad \forall \mathbf{u} \in \mathcal{N}, \tag{2.13}$$

So, Φ is bounded from below on $\mathcal{N}$. □

Lemma 2.1. *For every* $\mathbf{u} = (u, v) \in \mathbb{E}\setminus\{(0,0)\}$, *there is a number* $t > 0$ *such that* $t\mathbf{u} \in \mathcal{N}$.

Proof. For $(u, v) \in H^2(\mathbb{R}^N)\setminus\{(0,0)\}$ and $t > 0$, we have

$$\omega(t) := \Phi(tu, tv) = \frac{1}{2}t^2\|\mathbf{u}\|^2 - \frac{1}{3}t^3\int_{\mathbb{R}^N}(u^3 + v^3) - \frac{1}{2}\beta t^4\int_{\mathbb{R}^N} u^2v^2.$$

On the one hand, we have $\omega(0) = 0$ and $\omega(t) \ge C't^2$ for a small enough *t*. On the other hand, we have $\omega(t) \to -\infty$ as $t \to \infty$. So there is a maximum point t_m of *t*. Moreover we get $\omega'(t_m) = \Phi'(t_m\mathbf{u})\mathbf{u} = 0$ and deduce $t_m\mathbf{u} \in \mathcal{N}$. □

Lemma 2.2. *Assume that* $2 \le N \le 7$, *then* Φ *satisfies the PS condition constrained on* $\mathcal{N}$.

Proof. Suppose $\mathbf{u}_n = (u_n, v_n) \in \mathcal{N}$ is a sequence *i.e.*

$$\Phi(\mathbf{u}_n) \to c \quad \text{and} \quad \Phi'_{\mathcal{N}}(\mathbf{u}_n) \to 0, \quad \text{as } n \to \infty \tag{2.14}$$

From (2.4) and (2.9), we can get $\mathbf{u}_n$ is bounded, then we have a weakly convergent subsequence $\mathbf{u}_n \rightharpoonup \mathbf{u}_0 \in \mathbb{E}$ (for convenience denoted again by $\mathbf{u}_n$). Since H is compactly embedded into $L^P(\mathbb{R}^{\mathbb{N}})$ for $2 \le N \le 7$, we infer that

$$\int_{\mathbb{R}^N} u^3 \to \int_{\mathbb{R}^N} u_0^3, \int_{\mathbb{R}^N} v^3 \to \int_{\mathbb{R}^N} v_0^3, \int_{\mathbb{R}^N} u^2 \to \int_{\mathbb{R}^N} u_0^2, \int_{\mathbb{R}^N} u^2v^2 \to \int_{\mathbb{R}^N} u_0^2v_0^2.$$

Moreover, using the fact that $\mathbf{u}_n \in \mathcal{N}$ and (2.3), we have

$$\|\mathbf{u}\|^2 = \int_{\mathbb{R}^N} (u^3+v^3) + 2\beta\int_{\mathbb{R}^N} u^2v^2 \to \int_{\mathbb{R}^N} (u_0^3+v_0^3) + 2\beta\int_{\mathbb{R}^N} u_0^2v_0^2 \ge \rho.$$

which implies that $\mathbf{u}_0 \ne 0$. Letting

$$\Phi'_{\mathcal{N}}(\mathbf{u}) = \Phi'(\mathbf{u}) - \eta\Psi'(\mathbf{u}), \tag{2.15}$$

denotes the constrained gradient of Φ on $\mathcal{N}$ with $\eta \in \mathbb{R}$. Taking the scalar product with $\mathbf{u}_n \ne 0$ and with $\Phi'(\mathbf{u})[\mathbf{u}] = \Psi(\mathbf{u}) = 0$ we can get

$$\eta_n(\Psi'(\mathbf{u}_n, \mathbf{u}_n)) \to 0$$

then, taking into account (2.6) and (2.7), we deduce that η_n as $n \to 0$. We also have that $\|\Psi'(\mathbf{u}_n)\|$ is bounded, so with (2.13) and the fact η_n as $n \to 0$, we obtain

$$\|\Phi'(\mathbf{u}_n)\| \le \|\nabla_{\mathcal{N}}\Phi(\mathbf{u}_n)\| + |\eta_n|\|\Psi'(\mathbf{u}_n)\| \to 0, \quad \text{as } n \to \infty. \tag{2.16}$$

So we deduce that $\Psi'(\mathbf{u}_n) \to 0$ To finish the proof, since $\Phi'(\mathbf{u})[\mathbf{u}_0] \to 0$ as $n \to \infty$, it follows that $\mathbf{u}_n \to \mathbf{u}_0 \in \mathbb{E}$ strongly. □

Theorem 2.1. *Suppose* $\beta > \wedge^+$. *The infimum of* Φ *on* $\mathcal{N}$ *is attained at some* $\tilde{\mathbf{u}}$ *with* $\Phi(\tilde{\mathbf{u}}) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{v}_2)\}$ *and both components* $\tilde{u}, \tilde{v} \ne 0$.

Theorem 2.2.

1) *Let* $\beta_1 > 0$ *be the principal eigenvalue of*

$$\Delta^2\psi + \psi = \beta U_1^2\psi, \psi \in E,$$

and let $\psi_{\beta_1} > 0$ *be the corresponding positive eigenfunction. Then there exists* $\tau_0 > 0$ *such that when* $\beta \in (\beta_1 - \tau_0, \beta_1)$, (1.3) *has solutions* $(u_{1\beta}, v_{1\beta})$ *of the form*

$$u_{1\beta} = U_1 + o(\beta_1 - \beta),$$

and

$$\begin{aligned} v_{1\beta} &= s\psi_{\beta_1} + o(s) \\ &= \frac{\beta - \beta_1}{\beta'(0)}\psi_{\beta_1} + o(\beta_1 - \beta) \\ &= (\beta_1 - \beta)\frac{\int_{\mathbb{R}^N} U_1\psi_{\beta_1}^2}{\int_{\mathbb{R}^N} \psi_{\beta_1}^3}\psi_{\beta_1} + o(\beta_1 - \beta) \end{aligned}$$

2). *There exists* $\tau_1 > 0$ *such that when* $\beta \in (-\tau_1, \tau_1)$, (1.3) *has solutions* $(u_{2\beta}, v_{2\beta})$ *of the form*

$$u_{2\beta}=U_1+\beta\left(\Delta^2+\lambda_1-2U_1\right)^{-1}\left(U_1V_2^2\right)+o(\beta),$$
$$v_{2\beta}=V_2+\beta\left(\Delta^2+\lambda_2-2V_2\right)^{-1}\left(U_1^2V_2\right)+o(\beta).$$

3. Existence Results of Semi-Trivial Solutions and Non-Trivial Solutions

System (1.5) admits two kinds of semi-trivial solutions of the form $(u,0)$ and $(0,v)$. So we take $\mathbf{u}_1=(U_1,0)$ and $\mathbf{v}_2=(0,V_2)$, where U_1 and V_2 are radially symmetric ground state solutions of the equation $\Delta^2 w+\lambda_i w=w^2, i=1,2$ in $H^2(\mathbb{R}^N)$. Moreover, if we denote w a radially symmetric ground state solution of (3.1)

$$\Delta^2 w+w=w^2, \tag{3.1}$$

then, by scaling, we can get

$$U_1(x)=\lambda_1 w\left(\sqrt[4]{\lambda_1}x\right),\quad V_2(x)=\lambda_2 w\left(\sqrt[4]{\lambda_2}x\right), \tag{3.2}$$

Hence, system (1.5) has two kinds of semi-trivial solutions $\mathbf{u}_1=(U_1,0)$ and $\mathbf{v}_2=(0,V_2)$ with lowest energy among all the semi-trivial solutions.

Definition 3.1.

1) *We define new Nehari manifold corresponding to the equations of* (1.5) *by*

$$\mathcal{N}_1=\left\{u\in H^2(\mathbb{R}^N)\setminus\{0\}:J_1(0)=0\right\},$$
$$\mathcal{N}_2=\left\{v\in H^2(\mathbb{R}^N)\setminus\{0\}:J_2(v)=0\right\},$$

where

$$J_1(u):=I_1'(u)[u],\ J_2(v):=I_2'(v)[v].$$

Let us define the tangent space to $\mathcal{N}$ *on* $\mathbf{u}_1$ *and* $\mathbf{v}_2$ *by*

$$T_{\mathbf{u}_1}\mathcal{N}=\left\{\mathbf{h}\in H^2\times H^2:\Psi'(\mathbf{u}_1)[\mathbf{h}]=0\right\},$$
$$T_{\mathbf{v}_2}\mathcal{N}=\left\{\mathbf{h}\in H^2\times H^2:\Psi'(\mathbf{v}_2)[\mathbf{h}]=0\right\},$$

And define the tangent space to $\mathcal{N}_1$ *on* U_1 *and* $\mathcal{N}_2$ *on* V_2 *by*

$$T_{U_1}\mathcal{N}_1=\left\{h\in H^2(\mathbb{R}^N):J_1'(U_1)[h]=0\right\},$$
$$T_{V_2}\mathcal{N}_2=\left\{h\in H^2(\mathbb{R}^N):J_2'(V_2)[h]=0\right\}.$$

We can prove the following equivalence:

$$\mathbf{h}=(h_1,h_2)\in T_{\mathbf{u}_1}\mathcal{N}\Leftrightarrow h_1\in T_{U_1}\mathcal{N}_1,\quad \mathbf{h}=(h_1,h_2)\in T_{\mathbf{v}_2}\mathcal{N}\Leftrightarrow h_2\in T_{V_2}\mathcal{N}_2. \tag{3.3}$$

If we denote by $D^2\Phi_{\mathcal{N}}$ *the second derivative of* Φ *constrained on* $\mathcal{N}$, *using that* $\mathbf{v}_2$ *is a critical point of* Φ, *plainly we obtain that*

$$D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)[\mathbf{h}]^2=\Phi''(\mathbf{v}_2)[\mathbf{h}]^2,\quad \forall\mathbf{h}\in T_{\mathbf{v}_2}\mathcal{N}.$$

2) *We define the following Sobolev constants related to* U_1 *and* V_2,

$$S_1^2:=\inf_{\varphi\in E\setminus\{0\}}\frac{\|\varphi\|_2^2}{\int_{\mathbb{R}^N}U_1^2\varphi^2},\quad S_2^2:=\inf_{\varphi\in E\setminus\{0\}}\frac{\|\varphi\|_1^2}{\int_{\mathbb{R}^N}V_2^2\varphi^2}, \tag{3.4}$$

and

$$\wedge^{+}=\max\left\{S_1^2,S_2^2\right\},\ \wedge^{-}=\min\left\{S_1^2,S_2^2\right\}.$$

Proposition 3.1.

1) *If* $\beta<\wedge^{-}$ *then* $\mathbf{u}_1,\mathbf{v}_2$ *is a strict local minimum of* Φ *constrained on* $\mathcal{N}$ *.*

2) *If either* $\beta>\wedge^{+}$, *then* $\mathbf{u}_1,\mathbf{v}_2$ *is a saddle point of* Φ *constrained on* $\mathcal{N}$ *. Moreover*

$$\inf_{\mathcal{N}}\Phi(\mathbf{u})<\min\left\{\Phi(\mathbf{u}_1),\Phi(\mathbf{v}_2)\right\}, \tag{3.5}$$

Proof. 1) Suppose $\beta<S_2^2$.

For $\mathbf{h}\in T_{\mathbf{v}_2}\mathcal{N}$ one has that

$$D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)[\mathbf{h}]^2=\Phi''(\mathbf{v}_2)[\mathbf{h}]^2=\|h_1\|_1^2+I_2''(V_2)[h_2]^2-\beta\int_{\mathbb{R}^N}V_2^2h_1^2.$$

For one thing, since $\beta<S_2^2$ and the definition of S_2^2, there exists $c_1>0$ such that

$$\|h_1\|_1^2-\beta\int_{\mathbb{R}^N}V_2^2h_1^2\geq c_1\|h_1\|_1^2, \tag{3.6}$$

For another thing, using (3.3) and the fact that V_2 is a minimum of I_2 on $\mathcal{N}_2$, there exists a constant $c_2>0$ such that

$$I_2''(V_2)[h_2]^2\geq c_2\|h_2\|^2. \tag{3.7}$$

Hence, using (3.6) and (3.7) we get

$$D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)[\mathbf{h}]^2\geq c_1\|h_1\|_1^2+c_2\|h_2\|^2, \tag{3.8}$$

proving that V_2 is a strict local minimum of Φ on $\mathcal{N}$.

When $\beta<S_1^2$, we can obtain the same result by using the same argument as above.

2) Assume $\beta>S_2^2$

In this case, we choose an element $\tilde{h}_1\in H^2(\mathbb{R}^N)$, such that

$$S_2^2<\frac{\|\tilde{h}_1\|_1^2}{\int_{\mathbb{R}^N}\tilde{h}_1^2V_2^2}<\beta,$$

Now, taking $\tilde{\mathbf{h}}_1=(\tilde{h}_1,0)\in T_{\mathbf{v}_2}\mathcal{N}$ we get

$$D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)\left[\tilde{\mathbf{h}}_1\right]^2=\|\tilde{h}_1\|_1^2-\beta\int_{\mathbb{R}^N}V_2^2h_1^2<0,$$

And taking $\tilde{\mathbf{h}}_2=(0,\tilde{h}_2)\in T_{\mathbf{v}_2}\mathcal{N}$ we get

$$D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)\left[\tilde{\mathbf{h}}_2\right]^2=I_2''(V_2)\left[\tilde{h}_2\right]^2\geq c_2\|h_2\|_2^2.$$

Hence, V_2 is a saddle point of Φ on $\mathcal{N}$.

When $\beta>S_1^2$ we can obtain the same result using the same argument as above and obviously inequality (3.5) holds. □

Next, we will give the proof of Theorem 2.1 and Theorem 2.2.

Proof. By the Ekelands variational principle [26], there exists a minimizing sequence $\mathbf{u}_n \in \mathcal{N}$, *i.e.*,

$$\Phi(\mathbf{u}_n) \to c := \inf_{\mathcal{N}} \Phi, \quad \text{and} \quad \Phi'_{\mathcal{N}}(\mathbf{u}_n) \to 0.$$

Due to the Lemma 2.2, there exists $\tilde{\mathbf{u}} \in \mathcal{N}$ such that

$$\mathbf{u}_n \to \overline{\mathbf{u}} \quad \text{as} \quad n \to \infty.$$

so, $\overline{\mathbf{u}}$ is a minimum point of Φ on $\mathcal{N}$. □

We have $(\overline{u},\overline{v}) \in H^2 \times H^2$. Then there exists $t>0$ such that $(t|\overline{u}|, t|\overline{v}|) \in \mathcal{N}$. So we get

$$\|\overline{\mathbf{u}}\|^2 = t\int_{\mathbb{R}^N}\left(|\overline{u}|^3 + |\overline{v}|^3\right) + 2\beta t^2 \int_{\mathbb{R}^N} |\overline{u}|^2 |\overline{v}|^2.$$

Combining

$$\|\overline{\mathbf{u}}\|^2 = \int_{\mathbb{R}^N}\left(|\overline{u}|^3 + |\overline{v}|^3\right) + 2\beta \int_{\mathbb{R}^N} |\overline{u}|^2 |\overline{v}|^2,$$

we get $0 < t \le 1$. According to the definition of $\Phi_{\mathcal{N}}$,

$$\begin{aligned}\Phi(t|\overline{u}|, t|\overline{v}|) &= \frac{1}{6}t^2\|\overline{\mathbf{u}}\|^2 + \frac{1}{6}t^4\beta\int_{\mathbb{R}^N}\overline{u}^2\overline{v}^2 \\ &\le \frac{1}{6}\|\overline{\mathbf{u}}\|^2 + \beta\int_{\mathbb{R}^N}\overline{u}^2\overline{v}^2 \\ &= \Phi(\overline{u},\overline{v}) = c\end{aligned}$$

with $\Phi(t|\overline{u}|, t|\overline{v}|) \ge c$, we get $(u',v') := \Phi(t|\overline{u}|, t|\overline{v}|)$ is a nonnegative ground state solution of the system. We can conclude that both components of $\mathbf{u}'$ are non-trivial. In fact, if the second component $v' \equiv 0$, then $\mathbf{u}' = (u',0)$. So $\mathbf{u}' = (u',0)$ is the non-trivial solution of the system (1.5), Hence, we have

$$I_1(u') = \Phi(\mathbf{u}') < \Phi(\mathbf{u}_1) = I_1(U_1).$$

However, this is a contradiction due to the fact that U_1 is a radial ground state solution of $\Delta^2 u + \lambda u = u^2$. We conclude the first component $u' \ne 0$ using the same way. Lastly, taking into account Proposition 3.1-(2) and $\beta > \wedge^+$ we have

$$\Phi(\mathbf{u}') < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{v}_2)\}. \tag{3.9}$$

4. Bifurcation of Nontrivial Solutions

In this subsection, we prove the existence of nontrivial solution of (1.5) by using local bifurcation theory (see [27]). The main results follow.

Proof. Consider the eigenvalue problem

$$\Delta^2\psi + \psi = \beta U_1^2 \psi, \psi \in E, \tag{4.1}$$

It is well known that this problem admits a sequence of eigenvalues

$$0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots. \tag{4.2}$$

Moreover, we infer from [28] that the first eigenvalue $\beta_1 > 0$ is simple and the principle eigenfunction ψ_{β_1} is a positive function. Set
$\mathcal{S}^* = \{(\beta,u,v) = (\beta,U_1,0)\}$, We shall consider the bifurcation of nontrivial solution of (1.5) from the semitrivial branch $\mathcal{S}^*$ near $(\beta,U_1,0)$. To accomplish

this, we apply the bifurcation results of Crandall and Rabinowitz. First, we define F by

$$F(\beta,u,v)=\begin{pmatrix}\Delta^2 u+\lambda_1 u-u^2-\beta uv^2\\ \Delta^2 v+\lambda_2 v-v^2-\beta u^2 v\end{pmatrix} \tag{4.3}$$

Clearly, for $(\phi,\psi),(\phi_1,\psi_1),(\phi_2,\psi_2)\in E$, one sees that

$$F_{(u,v)}(\beta,u,v)[(\phi,\psi)]=\begin{pmatrix}\Delta^2\phi+\lambda_1\phi-2u\phi-\beta v^2\phi-2\beta uv\psi\\ \Delta^2\psi+\lambda_2\psi-2v\psi-2\beta uv\phi-\beta u^2\psi\end{pmatrix} \tag{4.4}$$

$$F_{(u,v)(u,v)}(\beta,u,v)[(\phi_1,\psi_1)(\phi_2,\psi_2)]=\begin{pmatrix}-2\phi_1\phi_2-2\beta\phi_1\psi_2 v-2\beta v\psi_1\phi_2-2\beta u\psi_1\psi_2\\ -2\psi_1\psi_2-2\beta u\psi_1\phi_2-2\beta v\phi_1\phi_2-2\beta u\phi_1\psi_2\end{pmatrix}$$

$$F_\beta(\beta,u,v)=\begin{pmatrix}-uv^2\\ -u^2 v\end{pmatrix}$$

$$F_{\beta(u,v)}(\beta,u,v)[(\phi,\psi)]=\begin{pmatrix}-v^2\phi-2uv\psi\\ -u^2\psi-2uv\phi\end{pmatrix}$$

We define

$$L(\phi,\psi)=F_{(u,v)}(\beta,U_1,0)[(\phi,\psi)]=\begin{pmatrix}\Delta^2\phi+\lambda_1\phi-2U_1\phi\\ \Delta^2\psi+\lambda_2\psi-\beta U_1^2\psi\end{pmatrix}=\begin{pmatrix}L_1(\phi)\\ L_2(\psi)\end{pmatrix} \tag{4.5}$$

From (4.1) and (4.2), we get that the null space $N(L_1)=span\{\psi_{\beta_1}\}$. The solution space of L_2 in $\mathbb{E}$ is $N_1=span\left\{\frac{\partial U_1^2}{\partial x_j}:2\le j\le N\right\}$. Hence, the null space $N(L_2)$ is trivial. So the null space $N(L)=span\{(\psi_{\beta_1},0)\}$, and ψ_{β_1} is the principal eigenfunction of (4.1). The range space of L is defined by

$$R(L)=\left\{(f,g)\in E:\int_{\mathbb{N}^N} g\psi_{\beta_1}=0\right\}. \tag{4.6}$$

Thus, $N(L)=codimR(L)=1$. Since $\int_{\mathbb{R}^N}U_1\psi_{\beta_1}>0$, it follows from (5.6) that

$$F_{\beta(u,v)}(\beta_1,U_1,0)[(0,\psi_{\beta_1})]=\begin{pmatrix}0\\ -U_1^2\psi_{\beta_1}\end{pmatrix}\ne R(L)$$

Thus, we can apply the result of [27] to conclude that the set of positive solutions to (1.5) near $(\beta_1,U_1,0)$ is a smooth curve

$$\Gamma=\left\{(\beta(s),u_{1\beta}(s),v_{1\beta}(s)):s\in(-\tau_0,\tau_0)\right\}, \tag{4.7}$$

such that $\beta(s)=\beta_1+\beta'(0)s+o(s)$, $u_{1\beta}(s)=U_1+o(s)$, $v_{1\beta}(s)=s\psi_{\beta_1}+o(s)$, where $\tau_0>0$ is a small constant. Moreover, $\beta'(0)$ can be calculated as (see, for example, [29] [30])

$$\beta'(0)=-\frac{\left\langle F_{(u,v)(u,v)}(\beta_1,U_1,0)[(0,\psi_{\beta_1})(0,\psi_{\beta_1})],\mathcal{L}\right\rangle}{2\left\langle F_{\beta(u,v)}(\beta_1,U_1,0)[(0,\psi_{\beta_1})],\mathcal{L}\right\rangle}=-\frac{\int_{\mathbb{R}^N}\psi_{\beta_1}^3}{\int_{\mathbb{R}^N}U_1\psi_{\beta_1}^2}, \tag{4.8}$$

where $\mathcal{L}$ is a linear functional on $\mathbb{E}$ defined as $\langle(f,g),\mathcal{L}\rangle=\int_{\mathbb{R}^N}g\psi_{\beta_1}$. Hence, we infer from (4.7) and (4.8) that for $\beta_1-\tau_0<\beta<\beta_1$

$$u_{1\beta} = U_1 + o(\beta_1 - \beta), \tag{4.9}$$

and

$$\begin{aligned} v_{1\beta} &= s\psi_{\beta_1} + o(s) \\ &= \frac{\beta - \beta_1}{\beta'(0)}\psi_{\beta_1} + o(\beta_1 - \beta) \\ &= (\beta_1 - \beta)\frac{\int_{\mathbb{R}^N} U_1 \psi_{\beta_1}^2}{\int_{\mathbb{R}^N} \psi_{\beta_1}^3}\psi_{\beta_1} + o(\beta_1 - \beta) \end{aligned}$$

Now, we give the proof of (2). As we know, $w_0 = (U_1(x), V_2(x))$ is the unique positive solution of (1.5) with $\beta = 0$. Recalling the map defined in (4.3), we have

$$F_{(u,v)}(0, U_1, V_2)[(\phi, \psi)] = \begin{pmatrix} \Delta^2\phi + \lambda_1\phi - 2U_1\phi \\ \Delta^2\psi + \lambda_2\psi - 2V_2\psi \end{pmatrix} \tag{4.10}$$

It is well known that $L_3 = \Delta^2 + \lambda_1 - 2U_1$ and $L_4 = \Delta^2 + \lambda_2 - 2V_2$ are both invertible; hence, w_0 is nondegenerate in X2r, *i.e.*, $F_{(u,v)}(0, U_1, V_2)$ exists. We infer from the implicit function theorem that there exists $\tau_2 > 0, R_0 > 0$ and $w_2(\beta): (-\tau_2, \tau_2) \to B_{R_0}(w_0)$ such that for any $\beta \in (-\tau_2, \tau_2)$, $F(\beta, w_2(\beta)) = F(\beta, u_{2\beta}, v_{2\beta}) = 0$. Moreover, we can compute (ϕ, ψ). Differentiating $F(\beta, U_1, V_2)$ by β at $\beta = 0$, because of $F(0, U_1, V_2) = 0$, we get

$$F_{(u,v)}(0, U_1, V_2)[(\phi, \psi)] = \begin{pmatrix} \Delta^2\phi + \lambda_1\phi - 2U_1\phi \\ \Delta^2\psi + \lambda_2\psi - 2V_2\psi - \end{pmatrix} = -\begin{pmatrix} -U_1V_2^2 \\ -U_1^2V_2 \end{pmatrix} \tag{4.11}$$

so

$$\phi = (\Delta^2 + \lambda_1 - 2U_1)^{-1}(U_1V_2^2), \psi = (\Delta^2 + \lambda_2 - 2V_2)^{-1}(U_1^2V_2) \tag{4.12}$$

This gives the expression of $(u_{2\beta}, v_{2\beta})$.

$$\begin{aligned} u_{2\beta} &= U_1 + \beta(\Delta^2 + \lambda_1 - 2U_1)^{-1}(U_1V_2^2) + o(\beta), \\ v_{2\beta} &= V_2 + \beta(\Delta^2 + \lambda_2 - 2V_2)^{-1}(U_1^2V_2) + o(\beta). \end{aligned}$$

5. Summary

In the paper, we studied the positive radial solutions for a higher order coupled system of Korteweg-de Vries equations in Theorem 2.1. Moreover, we proved the multiplicity of the equations by a bifurcation theory in Theorem 2.2.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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