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# Highly Efficient Method for Solving Parabolic PDE with Nonlocal Boundary Conditions

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## Abstract

In this work, a highly efficient algorithm is developed for solving the parabolic partial differential equation (PDE) with the nonlocal condition. For this purpose, we employ orthogonal Chelyshkov polynomials as the basis. The convergence analysis of the proposed scheme is derived. Numerical experiments are carried out to explain the efficiency and precision of the proposed scheme. Furthermore, the reliability of the scheme is verified by comparisons with assured existing methods.

## Keywords

Chelyshkov, Collocation Method, Parabolic, Nonlocal Boundary Conditions

## 1. Introduction

In the else decades, nonlocal boundary value problems have become a rapidly increasing field of research. The study of this type of problem is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics and life sciences can be modelled in this way. For example, problems with feedback controls such as the steady-states of a thermostat, transfer reactive and passive pollutants in underground water [1] [2], heat transfer, radioactive nuclear decay in fluid streams [3], viscoelastic material malformation in polymers [3], semiconductor modelling [4] and bioengineering.

The variety of physical phenomena developed on a (PDEs) concerning nonlocal integral terms is constantly increasing. The authors of [5] have given an example from metrology. This example is a prototype for the evolution of the system temperature distribution of air above the ground during calm nights. Specific problems occur in thermodynamics in thermoelasticity [6] [7] [8], heat transfer [9] [10] [11] and plasma physics [12]. The above-mentioned articles focus on the problems described in terms of parabolic equations. However, there are some problems dealing with the dynamics of the ground waters which are described in terms of hyperbolic equations [13].

The numerical research for PDEs with distinct types of non-local conditions is of great interest due to their broad range of applications. Several methods for solving nonlocal boundary problems have been developed such as finite-difference schemes [14], finite volume element method [15] [16], implicit finite difference scheme [17], Galerkin procedure [18] [19] [20], spectral collocation with preconditioning method [21], Chebyshev spectral collocation techniques [22] [23], Tau scheme [24], sinc method [25] [26] [27], sinc-Galerkin method [28], finite difference methods [14], spectral collocation method with preconditioning [21], Gaussian radial basis functions method [29], Legendre-Gauss-Lobatto pseudo-spectral method [30] [31]. Recently, El-Gamel and Abd El-Hady applied sinc collocation approach for solving a parabolic PDE with nonlocal boundary conditions [32]. Existence, uniqueness and some characteristics of the solution to these issues have been developed in [33].

In this paper, we attempt to introduce a new method, based on Chelyshkov polynomials for solving

$$\frac{\partial u(\eta,t)}{\partial t}=\frac{\partial^2 u(\eta,t)}{\partial \eta^2}+q(\eta,t),\quad 0<\eta<1,\quad 0<t\le 1 \tag{1}$$

with initial condition

$$u(\eta,0)=f(\eta) \tag{2}$$

subject to boundary conditions

$$\sum_{k=0}^{1}\alpha_k(t)\frac{\partial^k}{\partial \eta^k}u(0,t)+\int_0^1 k_1(\eta,t)u(\eta,t)\,\mathrm{d}\eta=g_1(t) \tag{3}$$

$$\sum_{k=0}^{1}\beta_k(t)\frac{\partial^k}{\partial \eta^k}u(1,t)+\int_0^1 k_2(\eta,t)u(\eta,t)\,\mathrm{d}\eta=g_2(t) \tag{4}$$

where $q, f, k_i, g_i, \alpha_i$ and $\beta_i$, $i=0,1$, are known functions.

The area of orthogonal polynomials is a very strong research area in mathematics as well as in purposes in mathematical physics, engineering and computer science. One of orthogonal polynomials is the set of the Chelyshkov polynomials. These polynomials have formed by Chelyshkov [34] [35], which are orthogonal over the interval $[0,1]$ with respect to the weight function $w(x)=1$. Chelyshkov orthogonal basis has been used for solving several kinds of integral and differential equations. For example, nonlinear weakly singular integral equations In [36], a class of mixed functional integro-differential equations In [37] [38], Volterra-Hammerstein delay integral equations In [39], the two-dimensional Fredholm-Volterra integral equations In [40], a systems of linear functional differential equations In [41] and distributed-order fractional differential equations In [42]. Moradi *et al.* In [43] applied Chelyshkov wavelets of time-delay fractional optimal control problems and El-Gamel *et al.* In [44] used Chelyshkov-

Tau approach for solving Bagley-Torvik equation.

The layout of this paper is as follows. Section 2, below briefly references, in which the reader can find an excellent summary of Chelyshkov polynomials which are required for establishing our results. Section 3 is dedicated to the formulation of Chelyshkov collocation scheme. In Section 4, the convergence analysis of the proposed method is investigated and error estimation for the fully discrete problem. Section 5 includes test examples to illustrate the accuracy and the performance of our scheme. Conclusion is made in Section 6.

## 2. An Overview and Relations of Chelyshkov Polynomials

An appropriate solution is expressed in the following form

$$u_N(\eta) \cong \sum_{j=0}^{N} a_j \psi_{N,j}(\eta)$$

so that $a_j$ and $\psi_{N,j}(\eta), j = 0,1,2,\cdots,N$, respectively, are the unknown Chelyshkov coefficients and Chelyshkov orthogonal polynomials of the degree $N$

$$\psi_{N,j}(\eta) = \sum_{i=0}^{N-j} (-1)^i \binom{N-j}{i} \binom{N+j+i+1}{N-j} \eta^{i+j}, \quad j = 0,1,\cdots,N \tag{5}$$

The Chelyshkov polynomials can be connected to Rodrigues' type expansion by

$$\psi_{N,j}(\eta) = \frac{1}{(N-j)!} \frac{1}{\eta^{j-1}} \frac{\mathrm{d}^{N-j}}{\mathrm{d}\eta^{N-j}} \left[ \eta^{N+j+1} (1-\eta)^{N-j} \right], \quad j = 0,1,\cdots,N$$

with orthogonality relation

$$\int_0^1 \psi_{N,i}(\eta) \psi_{N,j}(\eta) \mathrm{d}\eta = \begin{cases} \dfrac{1}{i+j+1}, & \text{for } i = j, \quad i,j = 0,1,\cdots,N,N+1 \\ 0 & \text{for } i \neq j. \end{cases} \tag{6}$$

Another relation to Chelyshkov polynomials from the previous one is

$$\int_0^1 \psi_{N,j}(\eta) \mathrm{d}\eta = \int_0^1 \eta^j \mathrm{d}\eta = \frac{1}{j+1}$$

The Chelyshkov polynomials could be represented in terms of the Jacobi polynomials $P_k^{(\alpha,\beta)}$ by the following relation

$$\psi_{N,j}(\eta) = (-1)^{N-j} \eta^{N-j} P_{N-j}^{(0,2j+1)}(2\eta - 1), \quad j = 0,1,\cdots,N.$$

We can write $u(\eta)$ and its derivative in matrix forms as follows:

$$\begin{aligned} u(\eta) &= \boldsymbol{\psi}(\eta) \boldsymbol{A} = \boldsymbol{\chi}(\eta) \boldsymbol{HA} \\ u^{(1)}(\eta) &= \boldsymbol{\psi}^{(1)}(\eta) \boldsymbol{A} = \boldsymbol{\chi}(\eta) \boldsymbol{MHA}, \\ u^{(2)}(\eta) &= \boldsymbol{\psi}^{(2)}(\eta) \boldsymbol{A} = \boldsymbol{\chi}(\eta) \boldsymbol{M}^2 \boldsymbol{HA}, \end{aligned} \tag{7}$$

where

$$\boldsymbol{\psi} = \left[ \psi_{N,0}, \psi_{N,1}, \cdots, \psi_{N,N} \right]$$

$$\boldsymbol{A} = \left[ a_0, \cdots, a_N \right]^{\tau}, \quad \text{and} \quad \boldsymbol{\chi}(\eta) = \left[ 1 \quad \eta \quad \eta^2 \quad \cdots \quad \eta^N \right]$$

for $N$ is odd,

$$H = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \cdots & 0 & 0 \\ -\binom{N}{1}\binom{N+2}{N} & \binom{N-1}{0}\binom{N+2}{N-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{N}{N-1}\binom{2N}{N} & -\binom{N-1}{N-2}\binom{2N}{N-1} & \cdots & \binom{1}{0}\binom{2N}{1} & 0 \\ -\binom{N}{N}\binom{2N+1}{N} & \binom{N-1}{N-1}\binom{2N+1}{N-1} & \cdots & -\binom{1}{1}\binom{2N+1}{1} & 1 \end{bmatrix}_{(N+1)\times(N+1)}$$

for $N$ is even,

$$H = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \cdots & 0 & 0 \\ -\binom{N}{1}\binom{N+2}{N} & \binom{N-1}{0}\binom{N+2}{N-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\binom{N}{N-1}\binom{2N}{N} & \binom{N-1}{N-2}\binom{2N}{N-1} & \cdots & \binom{1}{0}\binom{2N}{1} & 0 \\ \binom{N}{N}\binom{2N+1}{N} & -\binom{N-1}{N-1}\binom{2N+1}{N-1} & \cdots & -\binom{1}{1}\binom{2N+1}{1} & 1 \end{bmatrix}_{(N+1)\times(N+1)}$$

and

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

## 3. Direct Chelyshkov Collocation Method

In this part, we will approximate the solution of the Equation (1) as follows

$$u(\eta,t) \approx u_N(\eta,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{N,i}(\eta)\psi_{N,j}(t) = \psi(\eta,t)\boldsymbol{C}, \tag{8}$$

in which

$$\begin{aligned}\boldsymbol{\psi}(\eta,t) &= \boldsymbol{\psi}(\eta)\otimes\boldsymbol{\psi}(t) \\ &= \left[\psi_{N,0}(\eta)\psi_{N,0}(t)\cdots\psi_{N,0}(\eta)\psi_{N,N}(t)\cdots\psi_{N,N}(\eta)\psi_{N,0}(t)\cdots\psi_{N,N}(\eta)\psi_{N,N}(t)\right]\end{aligned}$$

where $\otimes$ represent for the kronecker product and also $\boldsymbol{C} = \left[c_{00}\cdots c_{0N}\cdots c_{N0}\cdots c_{NN}\right]^{\tau}$ which be an unknown vector and will be obtained by our scheme.

Clearly,

$$\begin{aligned}\frac{\partial}{\partial t}u_N(\eta,t)&=\left[\boldsymbol{\psi}(\eta)\otimes\boldsymbol{\psi}(t)\right]_t\boldsymbol{C}\\&=\left[\boldsymbol{\psi}(\eta)\otimes\left[\boldsymbol{\chi}(t)\boldsymbol{MH}\right]\right]\boldsymbol{C}\end{aligned}\tag{9}$$

Likewise, we should deduce that

$$\begin{aligned}\frac{\partial^2}{\partial\eta^2}u_N(x,t)&=\left[\boldsymbol{\psi}(\eta)\otimes\boldsymbol{\psi}(t)\right]_{\eta\eta}\boldsymbol{C}\\&=\left[\left[\boldsymbol{\chi}(\eta)\boldsymbol{M}^2\boldsymbol{H}\right]\otimes\boldsymbol{\chi}(t)\right]\boldsymbol{C}\end{aligned}\tag{10}$$

consider that $u(\eta,t)$ is approximated by $u_N(\eta,t)$ and we discretize the Equation (1) in the following form

$$\frac{\partial u_N(\eta_i,t_j)}{\partial t}=\frac{\partial^2 u_N(\eta_i,t_j)}{\partial\eta^2}+q(\eta_i,t_j),\ \ i=1,2,\cdots,N-1;\ j=1,2,\cdots,N\tag{11}$$

where

$$\eta_i=t_i=\frac{i}{N},\quad i=0,1,2,\cdots,N$$

by substituting (9) and (10) into Equation (11) we obtain

$$\begin{aligned}&\left[\boldsymbol{\psi}(\eta_i)\otimes\left[\boldsymbol{\chi}(t_j)\boldsymbol{MH}\right]-\left[\boldsymbol{\chi}(\eta_i)\boldsymbol{M}^2\boldsymbol{H}\right]\otimes\boldsymbol{\psi}(t_j)\right]\boldsymbol{C}\\&=q(\eta_i,t_j),\ \ i=1,2,\cdots,N-1;\ j=1,2,\cdots,N\end{aligned}\tag{12}$$

The initial conditions (2) can be discretized in the form

$$\boldsymbol{\psi}(\eta_i,0)\boldsymbol{C}=f(\eta_i),\quad i=0,1,\cdots,N\tag{13}$$

The boundary condition (3) can be discretized in the form

$$\boldsymbol{V}_1(t_j)\boldsymbol{C}=g_1(t_j),\quad j=1,2,\cdots,N\tag{14}$$

where

$$\begin{aligned}\boldsymbol{V}_1(t_j)=&\sum_{k=0}^{1}\alpha_k(t_j)\left[\boldsymbol{\chi}(0)\boldsymbol{M}^k\boldsymbol{H}\right]\otimes\boldsymbol{\psi}(t_j)\\&+\left[\int_0^1 k_1(\eta,t_j)\boldsymbol{\psi}(\eta)\mathrm{d}\eta\right]\otimes\boldsymbol{\psi}(t_j),\quad j=1,2,\cdots,N\end{aligned}$$

The boundary condition (4) is discretized in the form below

$$\boldsymbol{V}_2(t_j)\boldsymbol{C}=g_2(t_j),\quad j=1,2,\cdots,N\tag{15}$$

where

$$\begin{aligned}V_2(t_j)=&\sum_{k=0}^{1}\beta_k(t_j)\left[\boldsymbol{\chi}(1)\boldsymbol{M}^k\boldsymbol{H}\right]\otimes\boldsymbol{\psi}(t_j)\\&+\left[\int_0^1 k_2(\eta,t_j)\boldsymbol{\psi}(\eta)\mathrm{d}\eta\right]\otimes\boldsymbol{\psi}(t_j),\quad j=1,2,\cdots,N\end{aligned}$$

Equations (12)-(15) can be rewritten in the form

$$\boldsymbol{\Lambda C}=\boldsymbol{\Phi}\tag{16}$$

where

$$\Lambda = \begin{bmatrix} \psi(\eta_1)\otimes[\chi(t_1)MH]-[\chi(\eta_1)M^2H]\otimes\psi(t_1) \\ \vdots \\ \psi(\eta_1)\otimes[\chi(t_N)MH]-[\chi(\eta_1)M^2H]\otimes\psi(t_N) \\ \vdots \\ \psi(\eta_{N-1})\otimes[\chi(t_1)MH]-[\chi(\eta_{N-1})M^2H]\otimes\psi(t_1) \\ \vdots \\ \psi(\eta_{N-1})\otimes[\chi(t_N)MH]-[\chi(\eta_{N-1})M^2H]\otimes\psi(t_N) \\ \psi(\eta_0,0) \\ \vdots \\ \psi(\eta_N,0) \\ V_1(t_1) \\ \vdots \\ V_1(t_N) \\ V_2(t_1) \\ \vdots \\ V_2(t_N) \end{bmatrix}_{(N+1)^2\times(N+1)^2}$$

$$\Phi = \begin{bmatrix} q(\eta_1,t_1) \\ \vdots \\ q(\eta_1,t_N) \\ \vdots \\ q(\eta_{N-1},t_1) \\ \vdots \\ q(\eta_{N-1},t_N) \\ f(\eta_0) \\ \vdots \\ f(\eta_N) \\ g_1(t_1) \\ \vdots \\ g_1(t_N) \\ g_2(t_1) \\ \vdots \\ g_2(t_N) \end{bmatrix}_{(N+1)^2\times 1}$$

Then we solve the generated linear system of $(N+1)^2\times(N+1)^2$ equations by using Q-R method.

## 4. Convergence and Error Estimation

### 4.1. Convergence of Chelyshkov Polynomial

Now, we will introduce the convergence and error bound to Chelyshkov polynomials.

**Theorem 4.1.** *Suppose that the function* $u(\eta,t):[0,1]\times[0,1]\to\mathbb{R}^2$ *is* $k+1$

*times continuously differentiable,* $u(\eta,t)\in C^{k+1}[0,1]\times[0,1]$ *and* $\hat{u}(\eta,t)$ *is the best approximation of* $u(\eta,t)$ *in the space* $\mathbb{X}\times\mathbb{T}$ *where*

$$\mathbb{X}=Span\{\psi_{N,0}(\eta),\psi_{N,1}(\eta),\psi_{N,2}(\eta),\cdots,\psi_{N,N}(\eta)\}$$

and

$$\mathbb{T}=Span\{\psi_{N,0}(t),\psi_{N,1}(t),\psi_{N,2}(t),\cdots,\psi_{N,N}(t)\}$$

then the error bound is

$$\left\|u(\eta,t)-\hat{u}(\eta,t)\right\|_2\le\frac{M}{(k+1)!}\sqrt{\frac{2^{2k+4}-2}{(2k+3)(2k+4)}} \tag{17}$$

where, $\|.\|_2$ refers to the norm defined as

$$\left\|u(\eta,t)\right\|_2^2=\int_0^1\int_0^1\left|u(\eta,t)\right|^2\mathrm{d}\eta\mathrm{d}t,$$

and $M=\max\left|u^{(\vartheta)}(\eta,t)\right|,\vartheta=0,1,\cdots,k+1$

*Proof.* Due to $u(\eta,t)\in C^{k+1}[0,1]\times[0,1]$, then

$$\exists M>0:\forall(\eta,t)\in[0,1]\times[0,1],\left|u^{(\vartheta)}(\eta,t)\right|\le M,\vartheta=0,1,\cdots,k+1.$$

We use the Maclaurin expansion of $u(\eta,t)$ as follows:

$$u(\eta,t)=\sum_{\vartheta=0}^{k}\frac{1}{\vartheta!}\left(\eta\frac{\partial}{\partial\eta}+t\frac{\partial}{\partial t}\right)^{\vartheta}u(\eta,t)\Big|_{(\eta,t)=(0,0)}$$
$$+\frac{1}{(k+1)!}\left(\eta\frac{\partial}{\partial\eta}+t\frac{\partial}{\partial t}\right)^{k+1}u(\eta,t)\Big|_{(\eta,t)=(\hat{\eta},\hat{t})}$$

with $\hat{\eta},\hat{t}\in[0,1]$ and let

$$\tilde{u}(\eta,t)=\sum_{\vartheta=0}^{k}\frac{1}{\vartheta!}\left(\eta\frac{\partial}{\partial\eta}+t\frac{\partial}{\partial t}\right)^{\vartheta}u(\eta,t)\Big|_{(\eta,t)=(0,0)}$$

then the error bound

$$\begin{aligned}\left\|u(\eta,t)-\hat{u}(\eta,t)\right\|_2&\le\left\|u(\eta,t)-\tilde{u}(\eta,t)\right\|_2\\&=\left\|\frac{1}{(k+1)!}\left(\eta\frac{\partial}{\partial\eta}+t\frac{\partial}{\partial t}\right)^{k+1}u(\hat{\eta},\hat{t})\right\|_2\\&=\left(\int_0^1\int_0^1\left(\frac{1}{(k+1)!}\left(\eta\frac{\partial}{\partial\eta}+t\frac{\partial}{\partial t}\right)^{k+1}u(\hat{\eta},\hat{t})\right)^2\mathrm{d}\eta\mathrm{d}t\right)^{\frac{1}{2}}\end{aligned} \tag{18}$$

but

$$\left(\eta\frac{\partial}{\partial\eta}+t\frac{\partial}{\partial t}\right)^{k+1}u(\hat{\eta},\hat{t})=\sum_{i=0}^{k+1}\binom{k+1}{i}\eta^{k-i+1}t^{i}\frac{\partial^{k+1}}{\partial\eta^{k-i+1}\partial t^{i}}u(\eta,\hat{t})\le M(\eta+t)^{k+1} \tag{19}$$

by using two Equations (18), (19) we obtain

$$
\begin{aligned}
&\left(\int_0^1\int_0^1\left(\frac{1}{(k+1)!}\left(\eta\frac{\partial}{\partial\eta}+t\frac{\partial}{\partial t}\right)^{k+1}u\left(\hat{\eta},\hat{t}\right)\right)^2\mathrm{d}\eta\mathrm{d}t\right)^{\frac{1}{2}}\\
&\le\left(\int_0^1\int_0^1\left(\frac{M}{(k+1)!}(\eta+t)^{k+1}\right)^2\mathrm{d}\eta\mathrm{d}t\right)^{\frac{1}{2}}=\left(\int_0^1\int_0^1\left((\eta+t)^{k+1}\right)^2\mathrm{d}\eta\mathrm{d}t\right)^{\frac{1}{2}}\\
&=\frac{M}{(k+1)!}\left(\int_0^1\int_0^1(\eta+t)^{2k+2}\mathrm{d}\eta\mathrm{d}t\right)^{\frac{1}{2}}=\frac{M}{(k+1)!}\sqrt{\frac{2^{2k+4}-2}{(2k+3)(2k+4)}}
\end{aligned}
\tag{20}
$$

## 4.2. Error Estimation of Chelyshkov-Collocation Method

In this section, the error estimation for the Chelyshkov-collocation method has been employed with the residual error function [45] [46] [47] [48]. One can obtain the residual function first, we can display the residual function $R_N(\eta,t)$ as

$$R_N(\eta,t)=\hat{u}_t(\eta,t)-\hat{u}_{\eta\eta}(\eta,t)-q(\eta,t). \tag{21}$$

where $\hat{u}(\eta,t)$ is approximate solution given by (8) of Equation (1). Thus, $\hat{u}(\eta,t)$ fulfills the equation

$$
\begin{gathered}
\hat{u}_t(\eta,t)-\hat{u}_{\eta\eta}(\eta,t)=q(\eta,t)+R_N(\eta,t),\\
\hat{u}(\eta,0)=f(\eta)\\
\sum_{k=0}^{1}\alpha_k(t)\frac{\partial^k}{\partial\eta^k}\hat{u}(0,t)+\int_0^1 k_1(\eta,t)\hat{u}(\eta,t)\mathrm{d}\eta=g_1(t)\\
\sum_{k=0}^{1}\beta_k(t)\frac{\partial^k}{\partial\eta^k}\hat{u}(1,t)+\int_0^1 k_2(\eta,t)\hat{u}(\eta,t)\mathrm{d}\eta=g_2(t)
\end{gathered}
$$

so, we can obtain the error function

$$\varepsilon(\eta)=u(\eta,t)-\hat{u}(\eta,t), \tag{22}$$

such that $u(\eta,t)$ is the exact solution of Equation (1).

Accordingly, the error differential equation is

$$
\begin{aligned}
&\varepsilon_t(\eta,t)-\varepsilon_{\eta\eta}(\eta,t)=-R_N(\eta,t),\\
&\sum_{k=0}^{1}\alpha_k(t)\frac{\partial^k}{\partial\eta^k}\varepsilon(0,t)+\int_0^1 k_1(\eta,t)\varepsilon(\eta,t)\mathrm{d}\eta=0\\
&\sum_{k=0}^{1}\beta_k(t)\frac{\partial^k}{\partial\eta^k}\varepsilon(1,t)+\int_0^1 k_2(\eta,t)\varepsilon(\eta,t)\mathrm{d}\eta=0
\end{aligned}
\tag{23}
$$

The solution of Equation (23) is

$$\varepsilon(\eta,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}\tilde{c}_{ij}\psi_{N,i}(\eta)\psi_{N,j}(t)$$

In the same manner as Section 3, we obtain unknown coefficients $\tilde{c}_{ij}, i, j=0,1,2,\cdots,N$ . so, the maximum absolute error can be determined by

$$E_{\max}=\max\left|\varepsilon(\eta,t)\right|,\quad 0<\eta<1,\quad 0<t\le 1$$

Using maximum error estimation, we can test the reliability of the results especially if the exact solution is unknown.

## 5. Numerical Results

In this section, we experimentally illustrate the performance of the proposed scheme. Numerically, we verify that our proposed scheme can deal with the parabolic PDE equation with the nonlocal condition. We consider the following five examples, namely the nonlocal problems from [14] [21] [25] [29]. The formula of the maximum absolute error is

$$\|E_J\| = \max\left\{\left|u(\eta_i, t_j) - u_N(\eta_i, t_j)\right|\right\}, \quad i = 0,1,\cdots,N;\ j = 0,1,\cdots,N$$

**Example 1:** [14] consider the following parabolic PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \eta^2} - e^{-t}\left[\eta(\eta-1) + \frac{\delta^2}{6(1+\delta^2)} + 2\right]$$

subject to the boundary conditions

$$u(0,t) + \delta^2 \int_0^1 u(\eta,t)\,d\eta = 0$$

$$u(1,t) + \delta^2 \int_0^1 u(\eta,t)\,d\eta = 0$$

and the initial condition

$$u(\eta,0) = \eta(\eta-1) + \frac{\delta^2}{6(1+\delta^2)}$$

whose the exact solution is

$$u(\eta,t) = e^{-t}\left[\eta(\eta-1) + \frac{\delta^2}{6(1+\delta^2)}\right].$$

where $\delta = 0.12$ The maximum absolute error, $\|E_N\|$ is reported in **Table 1** as $N$ increases from $N = 3$ to $N = 9$. Maximum absolute error is tabulated in **Table 2** for Chelyshkov collocation with the analogous results of Ekolin [14] who used finite difference methods (Forward Euler, backward Euler and Crank-Nicolson methods) to obtain his numerical solution. The exact and approximating solutions and errors are shown in **Figure 1**.

**Example 2:** [25] consider the following

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \eta^2}$$

subject to the boundary conditions

$$u(0,t) - \int_0^1 (\eta + t) u(\eta,t)\,d\eta = g_1(t)$$

$$u(1,t) - \int_0^1 t e^{-\eta} u(\eta,t)\,d\eta = g_2(t)$$

where

$$g_1(t) = \frac{1}{2} - t - e^{-t}\left[\cos 1 + \sin 1 + t \sin 1 - 2\right]$$

**Table 1.** Maximum absolute error $\|E_N\|$ for example 1.

| $N$ | $\|E_N\|$ |
|---|---|
| 3 | 3.4117E-05 |
| 5 | 4.6525E-08 |
| 7 | 5.2898E-11 |
| 9 | 4.5264E-14 |

**Table 2.** Comparison of the numerical results for example 1.

| $\|E_N\|$, $N=9$ | 4.5264E-14 |
|---|---|
| Forward Euler $N=130$ [14] | 1.50E-07 |
| backward Euler, $N=130$ [14] | 6.70E-06 |
| Crank-Nicolson, $N=130$ [14] | 1.30E-06 |

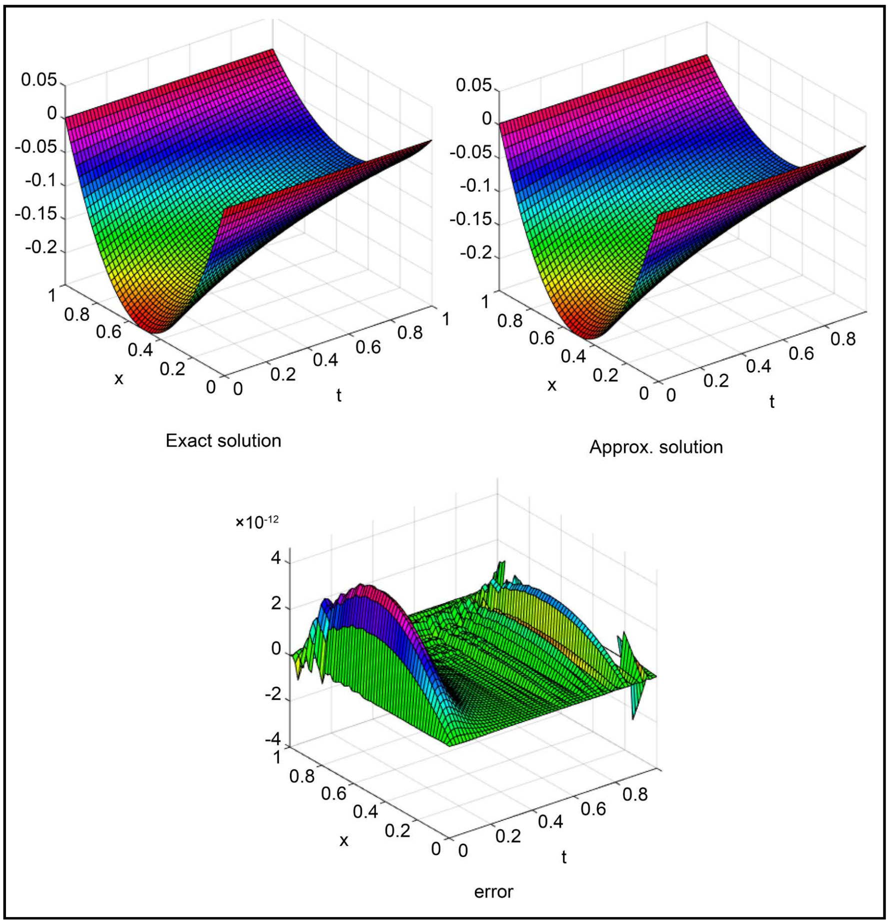


**Figure 1.** Exact, approximate solution and error for example 1.

$$g_2(t)=1+\mathrm{e}^{-t}\cos 1-\frac{t}{2\mathrm{e}}\left[2(\mathrm{e}-1)+\mathrm{e}^{-t}(\mathrm{e}-\cos 1+\sin 1)\right]$$

and the initial condition

$$u(\eta,0)=1+\cos(\eta)$$

whose the exact solution is

$$u(\eta,t)=1+e^{-t}\cos(\eta).$$

The maximum absolute error, $\|E_N\|$ is reported in **Table 3** as $N$ increases from $N=3$ to $N=9$. Maximum absolute error is tabulated in **Table 4** for Chelyshkov collocation method with the analogous results of Shidfar [25] who used sinc-collocation method to obtain this numerical solution. The exact and approximating solutions and error are shown in **Figure 2**.

**Example 3:** [21] [29] consider the following parabolic PDE

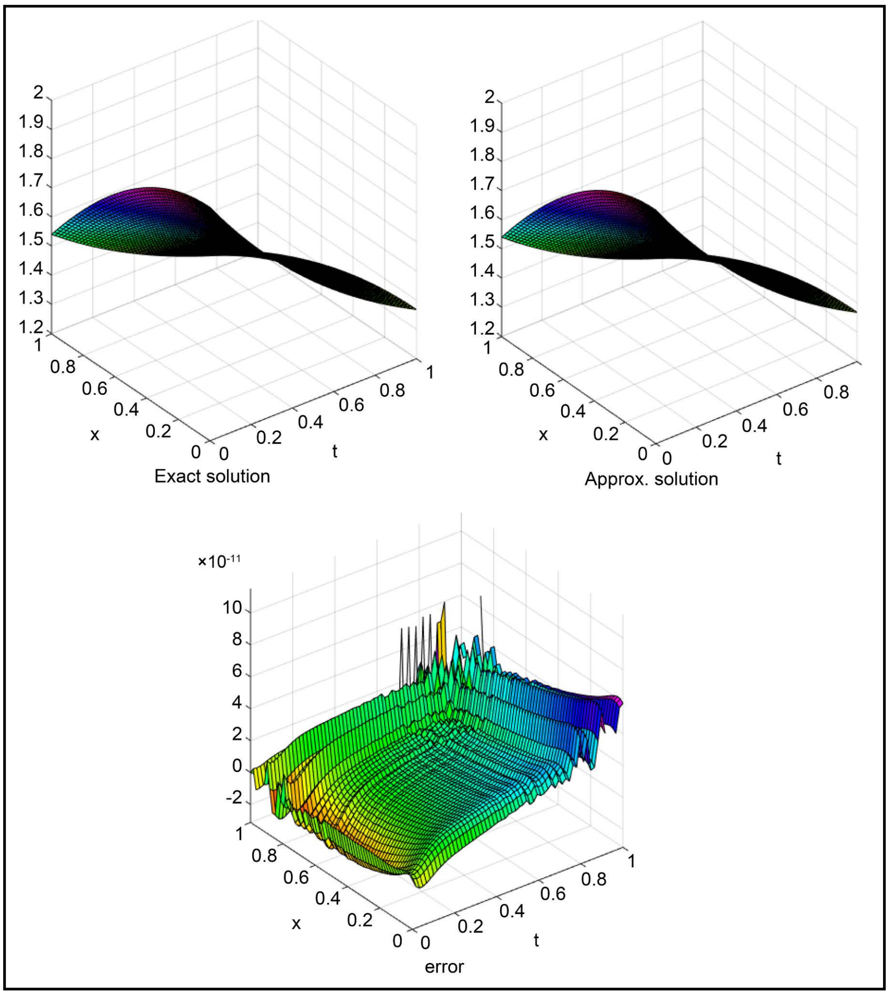


**Figure 2.** Exact, approximate solution and error for example 2.

**Table 3.** Maximum absolute error $\|E_N\|$ for example 2.

| $N$ | $\|E_N\|$ |
|---|---|
| 3 | 3.89250E-03 |
| 5 | 2.42266E-06 |
| 7 | 4.23364E-08 |
| 9 | 1.87322E-11 |

**Table 4.** Comparison of the numerical results for example 2.

| $\|E_N\|$, $N=9$ | Method [25], $N=30$ |
|---|---|
| 1.87E-11 | 1.83E-05 |

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial \eta^2},\quad 0<\eta<1,\quad 0<t<1$$

subject to the boundary conditions

$$u(0,t)=\mathrm{e}^{-\frac{\pi^2}{4}t}$$

$$\int_0^1 u(\eta,t)\,\mathrm{d}\eta=\frac{2}{\pi}\mathrm{e}^{-\frac{\pi^2}{4}t}$$

whose the exact solution is

$$u(\eta,t)=\exp\left(-\frac{\pi^2 t}{4}\right)\cos\left(\frac{\pi\eta}{2}\right).$$

In **Table 5** we display the comparison of absolute errors between our scheme and the absolute errors result from Gaussian radial method [29] at $t=0.1$. The exact and approximating solutions and errors are shown in **Figure 3**.

**Example 4:** [21] consider the following parabolic PDE

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial \eta^2}+\mathrm{e}^t\left(\eta^2-2\right),\quad 0<\eta<1,\quad 0<t<1$$

subject to the boundary conditions

$$u(0,t)=\eta^2$$

$$\int_0^1 u(\eta,t)\,\mathrm{d}\eta=\frac{\mathrm{e}^t}{3}$$

and the initial condition

$$u(\eta,0)=\eta^2$$

whose the exact solution is

$$u(\eta,0)=\mathrm{e}^t\eta^2.$$

In **Table 6**, we display the comparison of absolute errors between our scheme and the absolute errors in [21] at different values of $x$ and $t$. The exact and approximating solutions and error are shown in **Figure 4**.

**Example 5:** [29] consider the following parabolic PDE

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial \eta^2}+\left(\pi^2-1\right)\mathrm{e}^{-t}\left(\sin(\pi\eta)+\cos(\pi\eta)\right),\quad 0<\eta<1,\quad 0<t<1$$

subject to the boundary conditions

$$u(0,t)-2\int_0^1\sin(\pi\eta)u(\eta,t)\,\mathrm{d}\eta=0$$

$$u(0,t)+2\int_0^1\cos(\pi\eta)u(\eta,t)\,\mathrm{d}\eta=0$$

and the initial condition

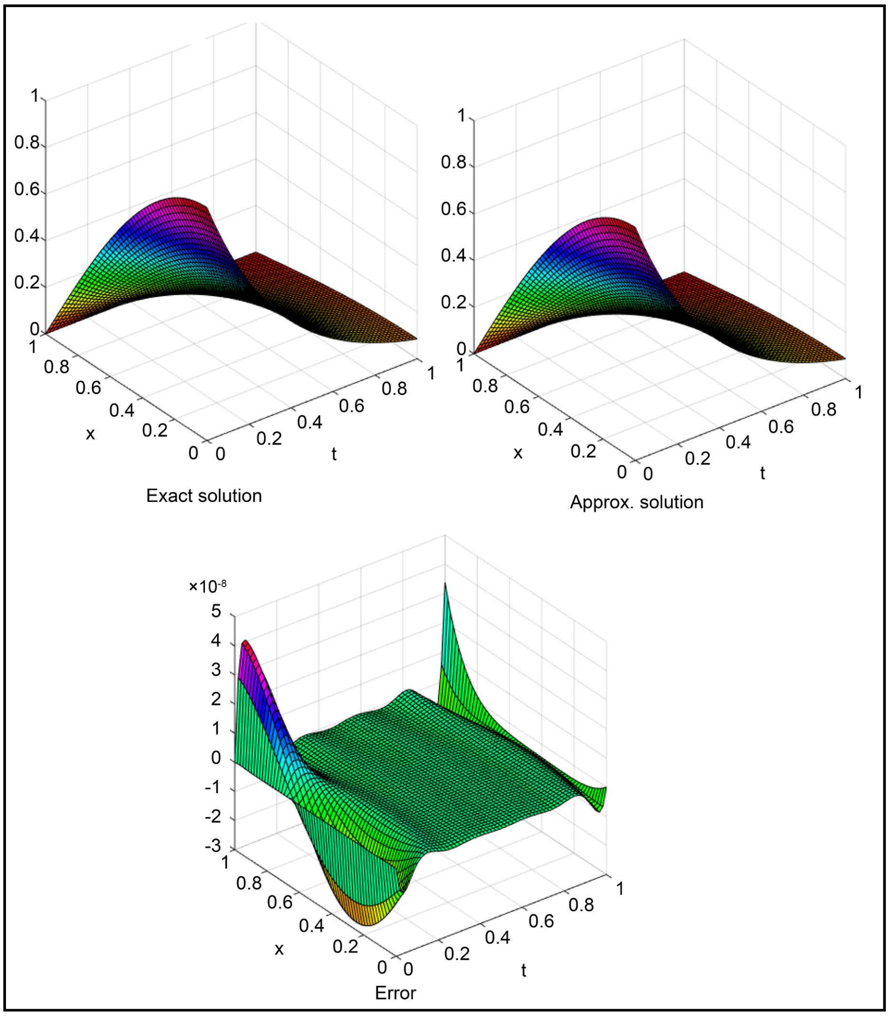


**Figure 3.** Exact, approximate solution and error for example 3.

**Table 5.** Absolute error with $t = 0.1$ and $N = 9$ for example 3.

| $x$ | our method | Method [29] |
|---|---|---|
| 0.1 | 7.6734E-09 | 2.6837E-08 |
| 0.2 | 1.2885E-08 | 3.0350E-08 |
| 0.3 | 1.4747E-08 | 2.3109E-08 |
| 0.4 | 1.3092E-08 | 1.1866E-08 |
| 0.5 | 8.2383E-09 | 1.0594E-09 |
| 0.6 | 1.0779E-09 | 3.7093E-09 |
| 0.7 | 7.0853E-09 | 7.5678E-09 |
| 0.8 | 1.4621E-08 | 5.5621E-08 |
| 0.9 | 1.9894E-08 | 1.7974E-07 |
| 1.0 | 2.0407E-08 | 4.5256E-07 |

**Table 6.** Comparison the absolute error of the our scheme with $N = 9$ and method in [21] for example 4.

| $(x,t)$ | our method | Method [21] |
|---|---|---|
| $(0.1,0.1)$ | 9.74824E-13 | 1.19E-08 |
| $(0.2,0.2)$ | 6.14349E-14 | 2.81E-11 |
| $(0.4,0.4)$ | 1.03907E-14 | 3.98E-11 |
| $(0.6,0.6)$ | 2.53647E-14 | 2.52E-11 |
| $(0.8,0.8)$ | 2.19862E-13 | 1.38E-13 |

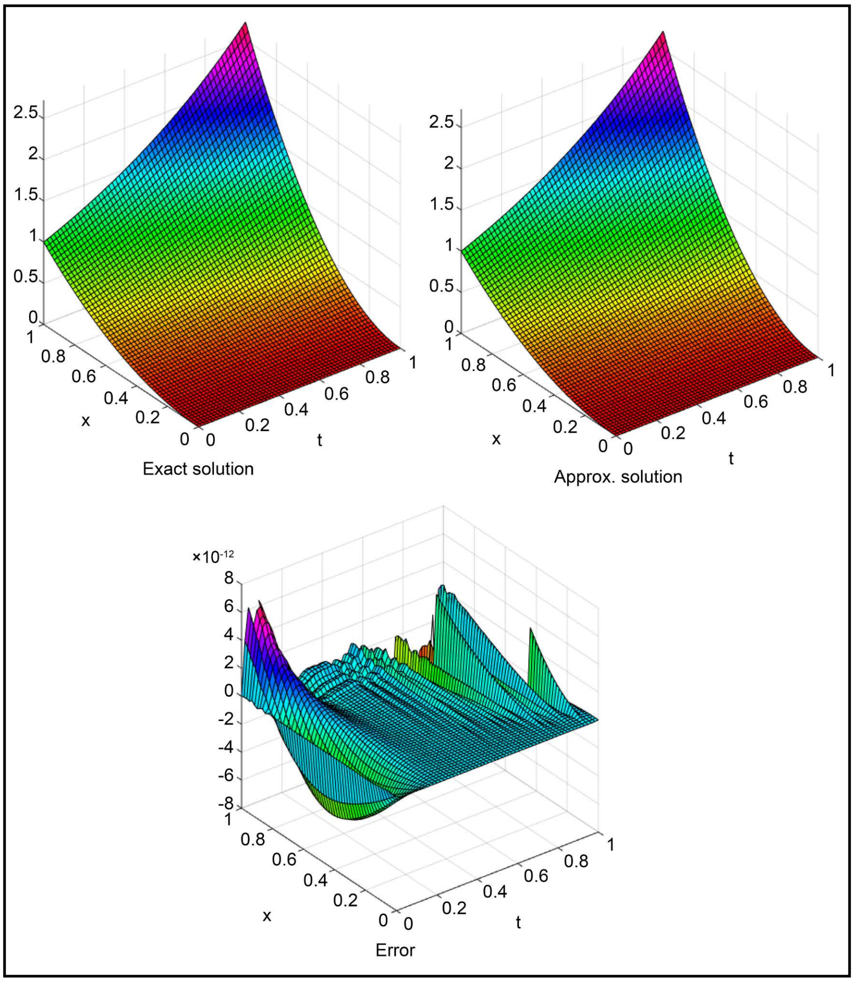


**Figure 4.** Exact, approximate solution and error for example 4.

$$u(\eta,0) = \sin(\pi\eta) + \cos(\pi\eta)$$

whose the exact solution is

$$u(\eta,t) = e^{-t}\left[\sin(\pi\eta) + \cos(\pi\eta)\right]$$

In **Table 7** we display the comparison of absolute errors between our scheme and the absolute errors result from Crank-Nicolson scheme [49] and Gaussian radial method [29] at $x = 0.25$. The exact and approximating solutions and errors are shown in **Figure 5**.

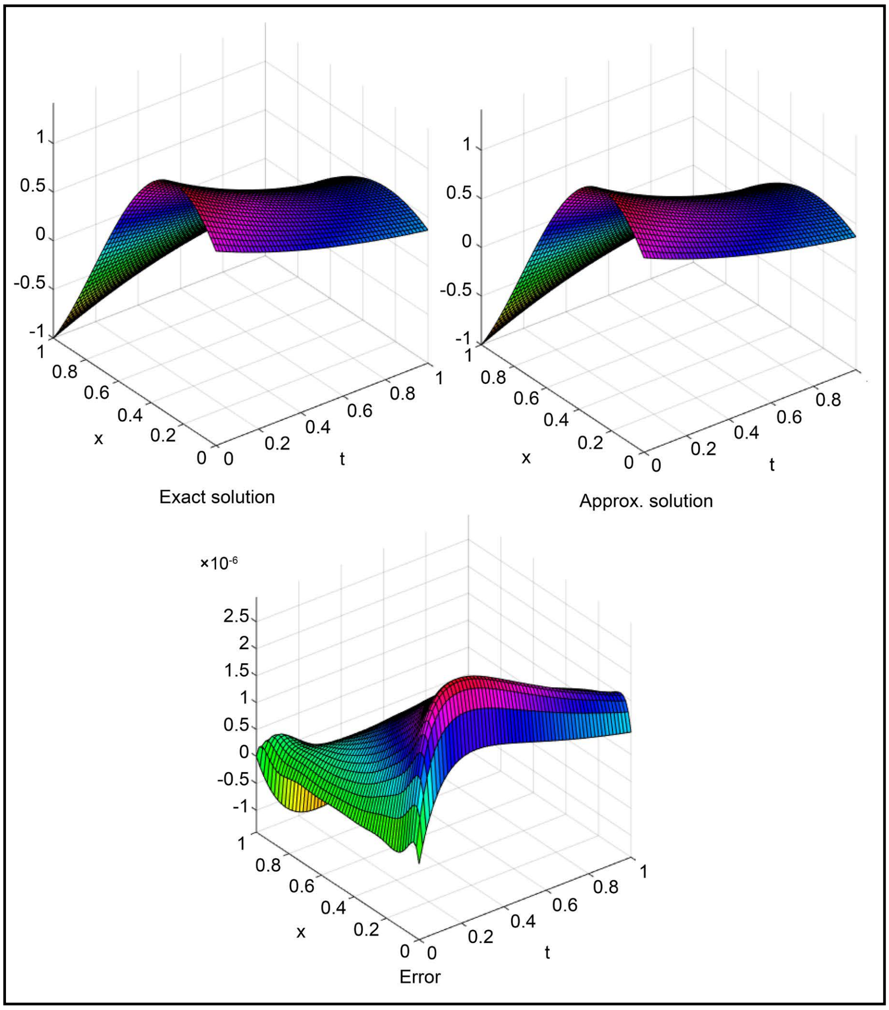


Figure 5. Exact, approximate solution and error for example 5.

Table 7. Absolute values of error with $x = 0.25$ and $N = 9$ for example 5.

| $t$ | present method | Method [49] | Method [30] |
|---|---|---|---|
| 0.1 | 1.5887E-06 | 5.17E-05 | 1.09E-06 |
| 0.2 | 2.1365E-06 | 6.19E-05 | 3.04E-06 |
| 0.3 | 2.2131E-06 | 6.49E-05 | 9.01E-06 |
| 0.4 | 2.1091E-06 | 6.45E-05 | 1.56E-06 |
| 0.5 | 1.9485E-06 | 6.21E-05 | 2.02E-06 |
| 0.6 | 1.6145E-06 | 5.64E-05 | 2.19E-06 |
| 0.7 | 1.6145E-06 | 4.99E-05 | 2.08E-06 |
| 0.8 | 1.4628E-06 | 4.49E-05 | 1.68E-06 |
| 0.9 | 1.3246E-06 | 4.08E-05 | 1.05E-06 |
| 1.0 | 1.1984E-06 | 3.64E-05 | 3.32E-06 |

## 6. Conclusion

This paper solved partial differential equations with nonlocal boundary conditions by applying Chelyshkov matrix collocation method. The numerical experiments demonstrated the efficiency of Chelyshkov matrix collocation method. In addition, the accuracy of the scheme was tested on five examples. The study found that the computational by Chelyshkov matrix collocation method can be an efficient numerical method to solve nonlocal problems.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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