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New Formula for Computing Quaternion Powers

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Abstract

In this work, we create a new mathematical formula that computes the power of a quaternion number raised to a positive integer by reducing the real matrix of order 4 × 4 that we take to represent this quaternion number to a matrix that makes the process of multiplying this quaternion number by itself simpler. We also present a new method for computing the power of a real matrix of order 2 × 2 as an application of this formula.

Keywords

Quaternions, Matrix Algebra, Kayley-Dickson Construction

1. Introduction

Quaternions have become very valuable in many fields such as computer graphics [1], physics [2], and engineering [3], one of the important motives for increasing quaternions applications is the development of quaternion algebra. The task of computing the power of a quaternion number in the simplest way is the aim of this work.

Applying the Cayley-Dickson construction [4] to two complex numbers $a = a_0 + a_1 i_1$ and $b = a_2 + a_3 i_1$ gives a convenient way to write any quaternion number q as:

$$\begin{aligned} q &= a + b i_2 \\ &= a_0 + a_1 i_1 + (a_2 + a_3 i_1) i_2 \\ &= a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 \end{aligned} \tag{1}$$

where a_0, a_1, a_2, a_3 are real numbers and i_1, i_2, i_3 are imaginary units satisfy Hamilton's rules [5] [6] [7], and their multiplication is given in **Table 1**.

Table 1. Imaginary units i_1, i_2, i_3 multiplication.

	i_1	i_2	i_3
i_1	-1	i_3	$-i_2$
i_2	$-i_3$	-1	i_1
i_3	i_2	$-i_1$	-1

The way of the multiplication operation of these units makes computing the powers of the quaternion number by using the binomial expansion is less used, especially when the quaternion number is raised to large power. De Moivre's and Euler's formulas [8] [9] are usually used to compute the power of a quaternion number.

Here, we give the simplest and most direct way to compute the power of a quaternion number raised to a positive integer n.

2. Methodology

Representing a quaternion number in the form of a matrix is very useful, especially in computing the powers of this number, there are 96 distinct real matrices [10] that represent a quaternion number, one of these matrices is a matrix A, which represents the number q.

$$A = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \tag{2}$$

And since:

$$A\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \omega\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

where:

$$\omega = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix} \tag{3}$$

we will use the matrix ω in computing q^n instead of using the matrix A, the matrix ω will play an important role in our computations.

Set:

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{4}$$

Then:

$$\begin{bmatrix} (A^2)_0 \\ (A^2)_1 \\ (A^2)_2 \\ (A^2)_3 \end{bmatrix} = \omega \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{5}$$

In general:

$$\begin{bmatrix} (A^n)_0 \\ (A^n)_1 \\ (A^n)_2 \\ (A^n)_3 \end{bmatrix} = \omega \begin{bmatrix} (A^{n-1})_0 \\ (A^{n-1})_1 \\ (A^{n-1})_2 \\ (A^{n-1})_3 \end{bmatrix} \tag{6}$$

We compute q^2, q^3, q^4, q^5 as:

$$\begin{bmatrix} (A^2)_0 \\ (A^2)_1 \\ (A^2)_2 \\ (A^2)_3 \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right) \\ 2a_0a_1 \\ 2a_0a_2 \\ 2a_0a_3 \end{bmatrix}$$

$$\begin{bmatrix} (A^3)_0 \\ (A^3)_1 \\ (A^3)_2 \\ (A^3)_3 \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right) \\ 2a_0a_1 \\ 2a_0a_2 \\ 2a_0a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_0^3 - 3a_0\left(a_1^2 + a_2^2 + a_3^2\right) \\ \left[3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right](a_1) \\ \left[3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right](a_2) \\ \left[3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right](a_3) \end{bmatrix}$$

$$\begin{bmatrix} (A^4)_0 \\ (A^4)_1 \\ (A^4)_2 \\ (A^4)_3 \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} a_0^3 - 3a_0\left(a_1^2 + a_2^2 + a_3^2\right) \\ \left[3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right](a_1) \\ \left[3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right](a_2) \\ \left[3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right](a_3) \end{bmatrix}$$

$$= \begin{bmatrix} a_0^4 - 6a_0^2\left(a_1^2 + a_2^2 + a_3^2\right) + \left(a_1^2 + a_2^2 + a_3^2\right)^2 \\ \left[4a_0^3 - 4a_0\left(a_1^2 + a_2^2 + a_3^2\right)\right](a_1) \\ \left[4a_0^3 - 4a_0\left(a_1^2 + a_2^2 + a_3^2\right)\right](a_2) \\ \left[4a_0^3 - 4a_0\left(a_1^2 + a_2^2 + a_3^2\right)\right](a_3) \end{bmatrix}$$

$$\begin{bmatrix} (A^5)_0 \\ (A^5)_1 \\ (A^5)_2 \\ (A^5)_3 \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} a_0^4 - 6a_0^2\left(a_1^2+a_2^2+a_3^2\right) + \left(a_1^2+a_2^2+a_3^2\right)^2 \\ \left[4a_0^3 - 4a_0\left(a_1^2+a_2^2+a_3^2\right)\right](a_1) \\ \left[4a_0^3 - 4a_0\left(a_1^2+a_2^2+a_3^2\right)\right](a_2) \\ \left[4a_0^3 - 4a_0\left(a_1^2+a_2^2+a_3^2\right)\right](a_3) \end{bmatrix}$$

$$= \begin{bmatrix} a_0^5 - 10a_0^3\left(a_1^2+a_2^2+a_3^2\right) + 5a_0\left(a_1^2+a_2^2+a_3^2\right)^2 \\ \left[5a_0^4 - 10a_0^2\left(a_1^2+a_2^2+a_3^2\right) + \left(a_1^2+a_2^2+a_3^2\right)^2\right](a_1) \\ \left[5a_0^4 - 10a_0^2\left(a_1^2+a_2^2+a_3^2\right) + \left(a_1^2+a_2^2+a_3^2\right)^2\right](a_2) \\ \left[5a_0^4 - 10a_0^2\left(a_1^2+a_2^2+a_3^2\right) + \left(a_1^2+a_1^2+a_3^2\right)^2\right](a_3) \end{bmatrix}$$

Now, we can conclude that when n is an even number, then:

$$\begin{bmatrix} (A^n)_0 \\ (A^n)_1 \\ (A^n)_2 \\ (A^n)_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n/2} \binom{n}{n-2i} a_0^{n-2i}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \\ a_1 \sum_{i=0}^{n/2-1} \binom{n}{n-2i-1} a_0^{n-2i-1}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \\ a_2 \sum_{i=0}^{n/2-1} \binom{n}{n-2i-1} a_0^{n-2i-1}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \\ a_3 \sum_{i=0}^{n/2-1} \binom{n}{n-2i-1} a_0^{n-2i-1}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \end{bmatrix} \tag{7}$$

and when n is an odd number, then:

$$\begin{bmatrix} (A^n)_0 \\ (A^n)_1 \\ (A^n)_2 \\ (A^n)_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{(n-1)/2} \binom{n}{n-2i} a_0^{n-2i}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \\ a_1 \sum_{i=0}^{(n-1)/2} \binom{n}{n-2i-1} a_0^{n-2i-1}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \\ a_2 \sum_{i=0}^{(n-1)/2} \binom{n}{n-2i-1} a_0^{n-2i-1}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \\ a_3 \sum_{i=0}^{(n-1)/2} \binom{n}{n-2i-1} a_0^{n-2i-1}\left[-\left(a_1^2+a_2^2+a_3^2\right)\right]^i \end{bmatrix} \tag{8}$$

Once we get $(A^n)_0, (A^n)_1, (A^n)_2, (A^n)_3$, then:

$$q^n = (A^n)_0 i_0 + (A^n)_1 i_1 + (A^n)_2 i_2 + (A^n)_3 i_3 \tag{9}$$

Special cases:

1) For a pure quaternion number $(a_0 = 0)$,

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} (A^2)_0 \\ (A^2)_1 \\ (A^2)_2 \\ (A^2)_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -(a_1^2 + a_2^2 + a_3^2) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (A^3)_0 \\ (A^3)_1 \\ (A^3)_2 \\ (A^3)_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -(a_1^2 + a_2^2 + a_3^2) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -(a_1^2 + a_2^2 + a_3^2)(a_1) \\ -(a_1^2 + a_2^2 + a_3^2)(a_2) \\ -(a_1^2 + a_2^2 + a_3^2)(a_3) \end{bmatrix}$$

$$\begin{bmatrix} (A^4)_0 \\ (A^4)_1 \\ (A^4)_2 \\ (A^4)_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -(a_1^2 + a_2^2 + a_3^2)(a_1) \\ -(a_1^2 + a_2^2 + a_3^2)(a_2) \\ -(a_1^2 + a_2^2 + a_3^2)(a_3) \end{bmatrix} = \begin{bmatrix} (a_1^2 + a_2^2 + a_3^2)^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (A^5)_0 \\ (A^5)_1 \\ (A^5)_2 \\ (A^5)_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (a_1^2 + a_2^2 + a_3^2)^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ (a_1^2 + a_2^2 + a_3^2)^2 (a_1) \\ (a_1^2 + a_2^2 + a_3^2)^2 (a_2) \\ (a_1^2 + a_2^2 + a_3^2)^2 (a_3) \end{bmatrix}$$

Therefore, if n is an even number, then the formula will be:

$$\begin{bmatrix} (A^n)_0 \\ (A^n)_1 \\ (A^n)_2 \\ (A^n)_3 \end{bmatrix} = \begin{bmatrix} \left[-(a_1^2 + a_2^2 + a_3^2)\right]^{n/2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

And if n is an odd number, then the formula will be:

$$\begin{bmatrix} (A^n)_0 \\ (A^n)_1 \\ (A^n)_2 \\ (A^n)_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (a_1)\left[-(a_1^2 + a_2^2 + a_3^2)\right]^{(n-1)/2} \\ (a_2)\left[-(a_1^2 + a_2^2 + a_3^2)\right]^{(n-1)/2} \\ (a_3)\left[-(a_1^2 + a_2^2 + a_3^2)\right]^{(n-1)/2} \end{bmatrix} \tag{12}$$

2) For a complex number $(a_2 = a_3 = 0)$, if n is an even number, then the formula will be:

$$\begin{bmatrix} (A^n)_0 \\ (A^n)_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n/2} \binom{n}{n-2i} (a_0^{n-2i}) (-a_1^2)^i \\ \left[\sum_{i-0}^{n/2-1} \binom{n}{n-2i-1} (a_0^{n-2i-1}) (-a_1^2)^i\right] (a_1) \end{bmatrix} \tag{13}$$

And if n is an odd number then the formula will be:

$$\begin{bmatrix} (A^n)_0 \\ (A^n)_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{(n-1)/2} \binom{n}{n-2i} (a_0^{n-2i})(-a_1^2)^i \\ \left[\sum_{i-0}^{(n-1)/2} \binom{n}{n-2i-1} (a_0^{n-2i-1})(-a_1^2)^i\right](a_1) \end{bmatrix} \tag{14}$$

These suitable formulas give the power of the quaternion number raised to n.

3. Examples

1) For $q = 1 + 2i_1 + 3i_2 + 4i_3$, let us compute q^5, q^{20}

Of course for small values of n, we can manually compute the power of the quaternion, and with helping of a computer program, we can compute the power of the quaternion when n is a large number.

To compute q^5, use (8),

$$(A^5)_0 = \sum_{i=0}^{2} \binom{5}{5-2i} (-29)^i$$

$$= 1 - 290 + 4205 = 3916$$

$$(A^5)_1 = 2\sum_{i=0}^{2} \binom{5}{4-2i} (-29)^i$$

$$= (2)(5 - 290 + 841) = (2)(556)$$

$$q^5 = 3916 + 556(2i_1 + 3i_2 + 4i_3)$$

To compute q^{20}, using the python code (**Algorithm 1**).

Set $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4$ and $n = 20$, the result will be:

a0 = −509333346214912.0

a1 = 110956955017216.0

a2 = 166435432525824.0

a3 = 221913910034432.0

$$q^{20} = -509333346214912 + 110956955017216 i_1 + 166435432525824 i_2 + 221913910034432 i_3$$

In the matrix form,

$$A = \begin{bmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & -4 & 3 \\ 3 & 4 & 1 & -2 \\ 4 & -3 & 2 & 1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 3916 & -1112 & -1668 & -2224 \\ 1112 & 3916 & -2224 & 1668 \\ 1668 & 2224 & 3916 & -1112 \\ 2224 & -1668 & 1112 & 3916 \end{bmatrix}$$

$$A^{20} = \begin{bmatrix} -509333346214912 & -110956955017216 & -166435432525824 & -221913910034432 \\ 110956955017216 & -509333346214912 & -221913910034432 & 166435432525824 \\ 166435432525824 & 221913910034432 & -509333346214912 & -110956955017216 \\ 221913910034432 & -166435432525824 & 110956955017216 & -509333346214912 \end{bmatrix}$$

```
def fact(n):
    res=1
    for i in range(2,n+1):
        res=res*i
    return res
def combi1(n,i):
    return (fact(n)/(fact(n-2*i)*fact(2*i)))
def combi2(n,i):
    return (fact(n)/(fact(n-2*i-1)*fact(2*i+1)))
def formu1(n,a0,a1,a2,a3):
    sum1=0
    for i in range(0,int(n/2+1)):
        sum1=sum1+combi1(n,i)*pow(a0,n-2*i)*pow(-(pow(a1,2)+pow(a2,2)+pow(a3,2)),i)
    return sum1
def formu2(n,a0,a1,a2,a3):
    sum2=0
    for i in range(0,int(n/2)):
        sum2=sum2+combi2(n,i)*pow(a0,n-2*i-1)*pow(-(pow(a1,2)+pow(a2,2)+pow(a3,2)),i)
    return sum2
#Driver code
n=?; a0=? ;a1=? ;a2=? ;a3=?
print("a0 =",formu1(n,a0,a1,a2,a3))
print("a1 =",a1*formu2(n,a0,a1,a2,a3))
print("a2 =",a2*formu2(n,a0,a1,a2,a3))
print("a3 =",a3*formu2(n,a0,a1,a2,a3))
```

Algorithm 1. Python code for computing q^n, n is an even integer.

2) To compute $\left(2i_1+3i_2+4i_3\right)^{101}$,

$$a_0=0, a_1=2, a_2=3 \text{ and } a_3=4$$

Using (12),

$$\begin{bmatrix} \left(A^{101}\right)_0 \\ \left(A^{101}\right)_1 \\ \left(A^{101}\right)_2 \\ \left(A^{101}\right)_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\left(29^{50}\right) \\ 3\left(29^{50}\right) \\ 4\left(29^{50}\right) \end{bmatrix}$$

$$\left(2i_1+3i_2+4i_3\right)^{101} = \left(29^{50}\right)\left(2i_1+3i_2+4i_3\right)$$

In the matrix form,

$$A = \begin{bmatrix} 0 & -2 & -3 & -4 \\ 2 & 0 & -4 & 3 \\ 3 & 4 & 0 & -2 \\ 4 & -3 & 2 & 0 \end{bmatrix}$$

$$A^{101} = \left(29^{50}\right)\begin{bmatrix} 0 & -2 & -3 & -4 \\ 2 & 0 & -4 & 3 \\ 3 & 4 & 0 & -2 \\ 4 & -3 & 2 & 0 \end{bmatrix}$$

4. Application

To compute the n[th] power of a matrix of order two, we will do a little change to our formulas to be compatible with matrices multiplication.

For $X=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we want to compute X^n.

Set $a_0 = a_{11}+a_{22}$, $a_1 = a_{11}-a_{22}$, $a_2 = a_{12}+a_{21}$, and $a_3 = a_{12}-a_{21}$, and:

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{15}$$

$$X^2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12}+a_{12}a_{22} \\ a_{21}a_{11}+a_{22}a_{21} & a_{21}a_{12}+a_{22}^2 \end{bmatrix}$$

$$\begin{bmatrix} (X^2)_0 \\ (X^2)_1 \\ (X^2)_2 \\ (X^2)_3 \end{bmatrix} = \begin{bmatrix} a_{11}^2 + 2a_{12}a_{21} + a_{22}^2 \\ a_{11}^2 - a_{22}^2 \\ a_{11}a_{12}+a_{11}a_{21}+a_{12}a_{22}+a_{21}a_{22} \\ a_{11}a_{12}-a_{11}a_{21} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a_0^2+a_1^2+a_2^2-a_3^2 \\ a_0a_1 \\ a_0a_2 \\ a_0a_3 \end{bmatrix}$$

$$\begin{aligned} 2X &= 2\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_0+a_1 & a_2+a_3 \\ a_2-a_3 & a_0-a_1 \end{bmatrix} \\ &= \begin{bmatrix} a_0 & 0 \\ 0 & a_1 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ a_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_3 \\ -a_3 & 0 \end{bmatrix} \end{aligned}$$

where $j_0=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $j_1=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $j_2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $j_3=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and their multiplication is given in **Table 2**.

Corresponding to the multiplication (**Table 2**), we can construct the matrix υ.

$$\upsilon = \begin{bmatrix} a_0 & a_1 & a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \tag{16}$$

And since:

$$\upsilon\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

We will reduce υ to υ'.

$$\upsilon' = \begin{bmatrix} a_0 & a_1 & a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix} \tag{17}$$

Notice that:

$$\upsilon'\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0^2 + a_1^2 + a_2^2 - a_3^2 \\ 2a_0a_1 \\ 2a_0a_2 \\ 2a_0a_3 \end{bmatrix}$$

Therefore, to compute X^2 we use:

$$\begin{bmatrix} (X^2)_0 \\ (X^2)_1 \\ (X^2)_2 \\ (X^2)_3 \end{bmatrix} = \frac{\upsilon'}{2}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{\upsilon'}{2}\begin{bmatrix} (X)_0 \\ (X)_1 \\ (X)_2 \\ (X)_3 \end{bmatrix}$$

Table 2. Matrices j_0, j_1, j_2, j_3 multiplication.

	j_0	j_1	j_2	$-j_3$
j_0	j_0	j_1	j_2	$-j_3$
j_1	j_1	j_0	j_3	$-j_2$
j_2	j_2	$-j_3$	j_0	j_1
j_3	j_3	$-j_2$	j_1	j_0

In general:

$$\begin{bmatrix} (X^n)_0 \\ (X^n)_1 \\ (X^n)_2 \\ (X^n)_3 \end{bmatrix} = \frac{\upsilon'}{2^n}\begin{bmatrix} (X^{n-1})_0 \\ (X^{n-1})_1 \\ (X^{n-1})_2 \\ (X^{n-1})_3 \end{bmatrix} \tag{18}$$

Therefore, to compute X^3 we use:

$$\begin{bmatrix} (X^3)_0 \\ (X^3)_1 \\ (X^3)_2 \\ (X^3)_3 \end{bmatrix} = \frac{\upsilon'}{8}\begin{bmatrix} (X^2)_0 \\ (X^2)_1 \\ (X^2)_2 \\ (X^2)_3 \end{bmatrix}$$

$$\begin{bmatrix} (X^3)_0 \\ (X^3)_1 \\ (X^3)_2 \\ (X^3)_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} a_0 & a_1 & a_2 & -a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix}\begin{bmatrix} a_0^2 + a_1^2 + a_2^2 - a_3^2 \\ 2a_0a_1 \\ 2a_0a_2 \\ 2a_0a_3 \end{bmatrix} = \begin{bmatrix} a_0^3 + 3a_0\left(a_1^2 + a_2^2 - a_3^2\right) \\ \left(3a_0^2 + a_1^2 + a_2^2 - a_3^2\right)(a_1) \\ \left(3a_0^2 + a_1^2 + a_2^2 - a_3^2\right)(a_2) \\ \left(3a_0^2 + a_1^2 + a_2^2 - a_3^2\right)(a_3) \end{bmatrix}$$

For computing X^4 we use:

$$\begin{bmatrix}(X^4)_0\\(X^4)_1\\(X^4)_2\\(X^4)_3\end{bmatrix} = \frac{1}{16}\begin{bmatrix} a_0 & a_1 & a_2 & -a_3\\ a_1 & a_0 & 0 & 0\\ a_2 & 0 & a_0 & 0\\ a_3 & 0 & 0 & a_0\end{bmatrix}\begin{bmatrix} a_0^3+3a_0\left(a_1^2+a_2^2-a_3^2\right)\\ \left(3a_0^2+a_1^2+a_2^2-a_3^2\right)(a_1)\\ \left(3a_0^2+a_1^2+a_2^2-a_3^2\right)(a_2)\\ \left(3a_0^2+a_1^2+a_2^2-a_3^2\right)(a_3)\end{bmatrix}$$

$$= \begin{bmatrix} a_0^4+6a_0^2\left(a_1^2+a_2^2+a_3^2\right)+\left(a_1^2+a_2^2+a_3^2\right)^2 \\ \left[4a_0^3+4a_0\left(a_1^2+a_2^2+a_3^2\right)\right](a_1)\\ \left[4a_0^3+4a_0\left(a_1^q+a_2^2+a_3^2\right)\right](a_2)\\ \left[4a_0^3+4a_0\left(a_1^2+a_2^2+a_3^2\right)\right](a_3)\end{bmatrix}$$

And for computing X^5 we use:

$$\begin{bmatrix}(X^5)_0\\(X^5)_1\\(X^5)_2\\(X^5)_3\end{bmatrix} = \frac{1}{32}\begin{bmatrix} a_0 & a_1 & a_2 & -a_3\\ a_1 & a_0 & 0 & 0\\ a_2 & 0 & a_0 & 0\\ a_3 & 0 & 0 & a_0\end{bmatrix}\begin{bmatrix} a_0^4+6a_0^2\left(a_1^2+a_2^2+a_3^2\right)+\left(a_1^2+a_2^2+a_3^2\right)^2 \\ \left[4a_0^3+4a_0\left(a_1^2+a_2^2+a_3^2\right)\right](a_1)\\ \left[4a_0^3+4a_0\left(a_1^q+a_2^2+a_3^2\right)\right](a_2)\\ \left[4a_0^3+4a_0\left(a_1^2+a_2^2+a_3^2\right)\right](a_3)\end{bmatrix}$$

$$= \begin{bmatrix} a_0^5+10a_0^3\left(a_1^2+a_2^2+a_3^2\right)+5a_0\left(a_1^2+a_2^2+a_3^2\right)^2 \\ \left[5a_0^4+10a_0^2\left(a_1^2+a_2^2+a_3^2\right)+\left(a_1^2+a_2^2+a_3^2\right)^2\right](a_1)\\ \left[5a_0^4+10a_0^2\left(a_1^q+a_2^2+a_3^2\right)+\left(a_1^2+a_2^2+a_3^2\right)^2\right](a_2)\\ \left[5a_0^4+10a_0^2\left(a_1^2+a_2^2+a_3^2\right)+\left(a_1^2+a_2^2+a_3^2\right)^2\right](a_3)\end{bmatrix}$$

We can conclude that when n is an even number, then:

$$\begin{bmatrix}(X^n)_0\\(X^n)_1\\(X^n)_2\\(X^n)_3\end{bmatrix} = \frac{1}{2^n}\begin{bmatrix} \sum_{i=0}^{n/2}\binom{n}{n-2i}a_0^{n-2i}\left(a_1^2+a_2^2-a_3^2\right)^i \\ a_1\sum_{i=0}^{n/2-1}\binom{n}{n-2i-1}a_0^{n-2i-1}\left(a_1^2+a_2^2-a_3^2\right)^i \\ a_2\sum_{i=0}^{n/2-1}\binom{n}{n-2i-1}a_0^{n-2i-1}\left(a_1^2+a_2^2-a_3^2\right)^i \\ a_3\sum_{i=0}^{n/2-1}\binom{n}{n-2i-1}a_0^{n-2i-1}\left(a_1^2+a_2^2-a_3^2\right)^i \end{bmatrix} \quad (19)$$

and when n is an odd number, then:

$$\begin{bmatrix}(X^n)_0\\(X^n)_1\\(X^n)_2\\(X^n)_3\end{bmatrix} = \frac{1}{2^n}\begin{bmatrix} \sum_{i=0}^{(n-1)/2}\binom{n}{n-2i}a_0^{n-2i}\left(a_1^2+a_2^2-a_3^2\right)^i \\ a_1\sum_{i=0}^{(n-1)/2}\binom{n}{n-2i-1}a_0^{n-2i-1}\left(a_1^2+a_2^2-a_3^2\right)^i \\ a_2\sum_{i=0}^{(n-1)/2}\binom{n}{n-2i-1}a_0^{n-2i-1}\left(a_1^2+a_2^2-a_3^2\right)^i \\ a_3\sum_{i=0}^{(n-1)/2}\binom{n}{n-2i-1}a_0^{n-2i-1}\left(a_1^2+a_2^2-a_3^2\right)^i \end{bmatrix} \quad (20)$$

Once we get $(X^n)_0, (X^n)_1, (X^n)_2, (X^n)_3$ then:

$$X^n = (X^n)_0 j_0 + (X^n)_1 j_1 + (X^n)_2 j_2 + (X^n)_3 j_3 \tag{21}$$

Special case:

When $a_0 = 0$, if n is an even number, the formula will be:

$$\begin{bmatrix} (X^n)_0 \\ (X^n)_1 \\ (X^n)_2 \\ (X^n)_3 \end{bmatrix} = \frac{1}{2^n} \begin{bmatrix} (a_1^2 + a_2^2 - a_3^2)^{n/2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

And if n is an odd number, the formula will be:

$$\begin{bmatrix} (X^n)_0 \\ (X^n)_1 \\ (X^n)_2 \\ (X^n)_3 \end{bmatrix} = \frac{1}{2^n} \begin{bmatrix} 0 \\ a_1 (a_1^2 + a_2^2 - a_3^2)^{(n-1)/2} \\ a_2 (a_1^2 + a_2^2 - a_3^2)^{(n-1)/2} \\ a_3 (a_1^2 + a_2^2 - a_3^2)^{(n-1)/2} \end{bmatrix} \tag{23}$$

Therefore, it is easy to compute the power of a matrix of order two raised to n.

5. Examples

1) For $X = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, let us compute X^{11}, X^{100}

To compute X^{11}, use (20).

Set $a_0 = a_{11} + a_{22} = -2, a_1 = a_{11} - a_{22} = 4, a_2 = a_{12} + a_{21} = 6, a_3 = a_{12} - a_{21} = -2$

$$\begin{bmatrix} (X^{11})_0 \\ (X^{11})_1 \\ (X^{11})_2 \\ (X^{11})_3 \end{bmatrix} = \frac{1}{2^{11}} \begin{bmatrix} \sum_{i=0}^{5} \binom{11}{11-2i} (4^{11-2i})(36^i) \\ -2\sum_{i=0}^{4} \binom{11}{10-2i} (4^{10-2i})(36^i) \\ 6\sum_{i=0}^{4} \binom{11}{10-2i} (4^{10-2i})(36^i) \\ -2\sum_{i=0}^{4} \binom{11}{10-2i} (4^{10-2i})(36^i) \end{bmatrix}$$

$$\begin{aligned} (X^{11})_0 &= \frac{1}{2^{11}} \big[4^{11} + (55)(4^9)(36) + (330)(4^7)(36^2) + (462)(4^5)(36^3) \\ &\quad + (165)(4^3)(36^4) + (11)(4)(36^5) \big] \\ &= 24414062 \end{aligned}$$

$$\begin{aligned} (X^{11})_1 &= \frac{1}{2^{11}} \big[(11)(4^{10}) + (165)(4^8)(36) + (462)(4^6)(36^2) + (330)(4^4)(36^3) \\ &\quad + (55)(4^2)(36^4) + 36^5 \big](-2) \\ &= (4069010.5)(-2) = -8138021 \end{aligned}$$

$$\left(X^{11}\right)_2 = (4069010.5)(6) = 24414063$$

$$\left(X^{11}\right)_3 = (4069010.5)(-2) = -8138021$$

$$X^{11} = (24414062)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - (8138021)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$+(24414063)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - (8138021)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 16276041 & 16276042 \\ 32552084 & 32552083 \end{bmatrix}$$

To compute X^{100}, Let us use the python code (Algorithm 2).

Set $a_{11} = 1, a_{12} = 2, a_{21} = 4, a_{22} = 3$ and $n = 100$, the result will be:

a11 = 2.6295363507367063e+69

a12 = 2.6295363507367063e+69

a21 = 5.2590727014734134e+69

a22 = 5.2590727014734134e+69

2) For $X = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$, let us compute X^{100}

Set $a_0 = 0, a_1 = 2, a_2 = 6, a_3 = -2$, and use (21). We will get:

```
def fact(n):
    res=1
    for i in range (2, n+1):
        res=res*i
    return res
def combi1(n,i):
    return (fact(n)/(fact(n-2*i)*fact(2*i)))
def combi2(n,i):
    return (fact(n)/(fact(n-2*i-1)*fact(2*i+1)))
def formu1(n,a0,a1,a2,a3):
    sum1=0
    for i in range (0,int(n/2+1)):
        sum1=sum1+combi1(n,i)*pow(a0,n-2*i)*pow((pow(a1,2)+pow(a2,2)-pow(a3,2)),i)
    return sum1
def formu2(n,a0,a1,a2,a3):
    sum2=0
    for i in range (0,int(n/2)):
        sum2=sum2+combi2(n,i)*pow(a0,n-2*i-1)*pow((pow(a1,2)+pow(a2,2)-pow(a3,2)),i)
    return sum2
# Driver code
n=? ; a11=? ; a12=? ; a21=? ; a22=?
a0=a11+a22;a1=a11-a22;a2=a12+a21;a3=a12-a21
print("a11 =",(formu1(n,a0,a1,a2,a3)+a1*formu2(n,a0,a1,a2,a3))/pow(2,n))
print("a12 =",(a2*formu2(n,a0,a1,a2,a3)+a3*formu2(n,a0,a1,a2,a3))/pow(2,n))
print("a21 =",(a2*formu2(n,a0,a1,a2,a3)-a3*formu2(n,a0,a1,a2,a3))/pow(2,n))
print("a22 =",(formu1(n,a0,a1,a2,a3)-a1*formu2(n,a0,a1,a2,a3))/pow(2,n))
```

Algorithm 2. Python code for computing A^n, n is an even integer.

$$\left(X^{100}\right)_0 = \frac{1}{2^{100}}(36)^{50}$$

$$X^{100} = 3^{100}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6. Conclusion

Representing the quaternion number q in a matrix form like A and reducing this matrix to the matrix ω make multiplying q by itself easier, and this makes creating the formula for computing the n^{th} power of q be very simple. The paper followed this way and created an important formula for computing q^n and applied a similar formula to compute the power of a matrix of order two.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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